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Predicting the Sustainability of Accelerating Magnetic Gear Trains

How long can a magnetic gear train can be accelerated at a given rate before slipping occurs, given the moment of inertia of the driven load?

Extended Essay in Physics

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1 Introduction

Magnetic gears are intriguing. How can two objects that are not even physically connected transfer torque effectively?

Well, just how effectively *can* they transfer torque?

1.1 Research Question

The research question for this essay is *how long can a magnetic gear train can be accelerated at a given rate before slipping occurs, given the moment of inertia of the driven load?* Knowing this amount of time reveals the sustainability of the acceleration being analyzed.

1.2 Magnetic Gears

Magnetic gears are gears whose teeth are replaced with magnetic fields: magnets of alternating polarities are evenly distributed along the perimeter of the gearwheel, and the fields of the magnetic "teeth" of each gear interlock.

Magnetic gear trains are systems of magnetic gears whose magnetic fields interlock with those of adjacent gears. This essay will focus on magnetic gear trains with two gears: a "driver" gear and a "driven" gear. The driver gear delivers torque to the gear train from an external source (e.g., a motor). The driven gear is acted upon by the driver gear and transfers torque to a load. An "accelerating" magnetic gear train is a gear train whose driver gear is accelerating, which accelerates the driven gear in turn.

1.2.1 Design of Magnetic Gears

I designed custom magnetic gears for use in this exploration. My design was inspired by that of John Atkinson from his YouTube video *Magnetic Gears* (2015).

Essentially, I evenly distribute eight strong neodymium magnets along the perimeter of a 3D printed frame. The magnets alternate in polarity so as to truly *interlock* with each other's magnetic fields. An even number of magnets needs to be used to ensure smooth transmission of torque, and so as not to have large gaps between each magnet as would occur with, say, only four or six magnets, eight magnets are used. More than eight magnets would be too expensive and potentially compromise the strength of the frame. N42 (1.32 T) neodymium disc magnets $0.0250\,\mathrm{m} \pm 0.0001\,\mathrm{m}$ in diameter and $0.0100\,\mathrm{m} \pm 0.0001\,\mathrm{m}$ in height are used[1].

In total, two identical gears are assembled and fixed onto their own axles.

[1]The magnets were purchased from Indigo Instruments (SKU 44209-10)

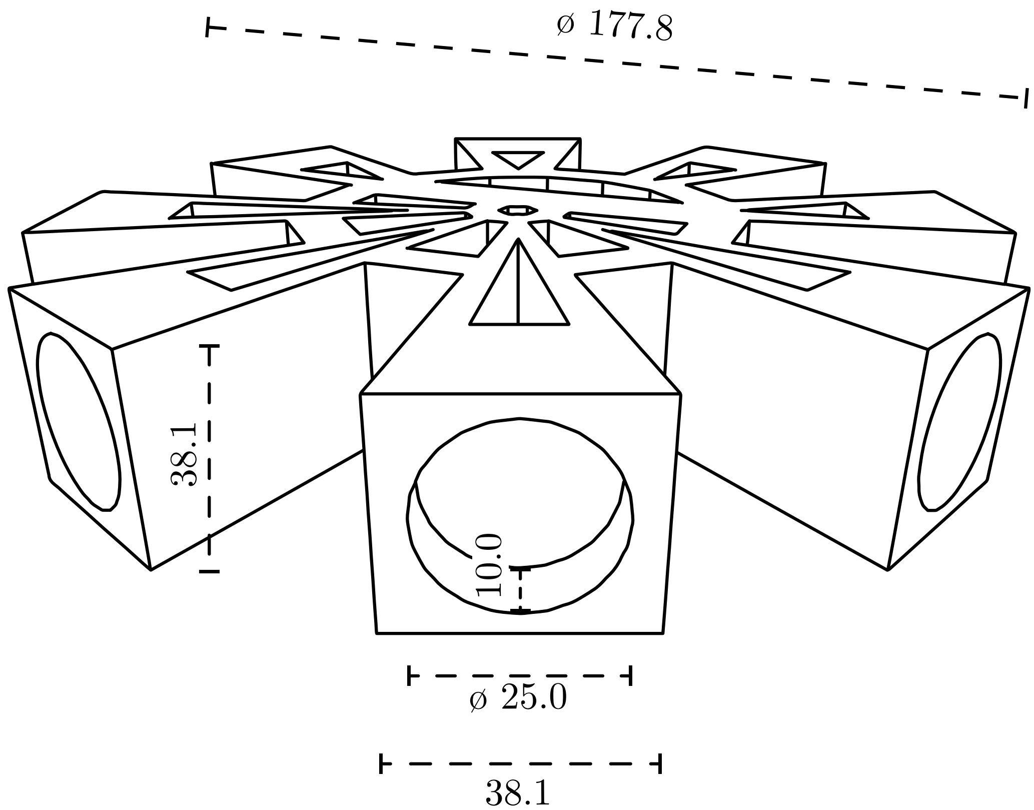


Figure 1: The 3D printed frame of the magnetic gear I designed. All measurements[2] are in mm ± 0.1 mm. See Appendix A for my design process.

1.2.2 Work Performed by Magnetic Gears

Figure 2 illustrates how, as the driver gear rotates and one of its magnets begins to attract a magnet on the driven gear, another pair of magnets is inevitably separated. This demonstrates that magnetic gears cannot introduce energy to a system: for every joule of potential energy converted to kinetic energy as the magnets approach, one joule of kinetic energy is converted back to one joule of potential energy as the magnets separate again.

[2]In this essay, measurements on diagrams will be provided in mm to enhance readability. Otherwise, the Standard International base unit of m is preferred.

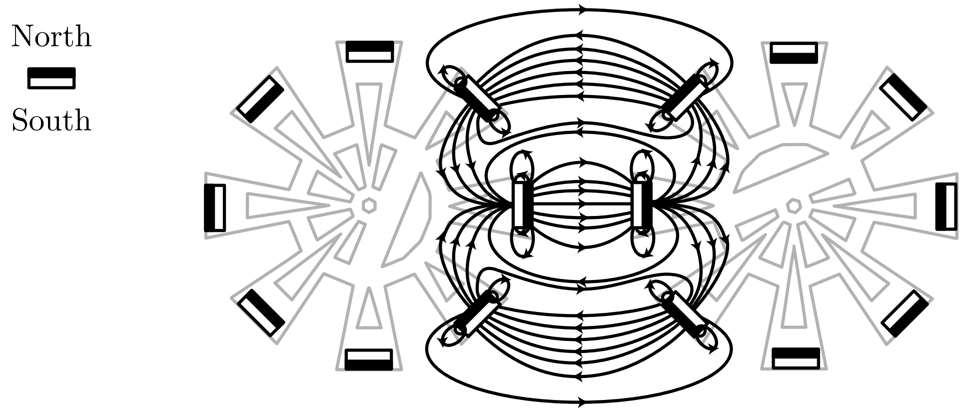


Figure 2: A simplified illustration of the magnetic fields between two aligned magnetic gears with magnets arranged with alternating polarities.

1.3 Hypothesyzing Torque Transfer

Newton's first law of motion tells us that a torque must act on the driven gear in order for it to accelerate from rest. This torque comes from the magnetic fields of the accelerating driver gear interacting with those of the driven gear.

Because the gears in magnetic gear trains do not interlock physically, they can accelerate at different rates. The driver gear accelerates at the same rate $\vec{\alpha}_i$ as its source of torque. The driven gear will accelerate at a rate $\vec{\alpha}_o$ determined in part by the torque applied onto the driven gear from the driver gear's magnets.

I hypothesize that the torque between two magnetic gears is a function of the difference in their relative angular displacements.

I think this because a magnetic dipole with a magnetic dipole moment $\vec{\mu}$ placed in a magnetic field $\overrightarrow{B}$ has a potential energy E_p equivalent to the dot product of these two vectors (Acosta, 2006):

$$E_p = \vec{\mu} \cdot \overrightarrow{B} = \|\vec{\mu}\|\|\overrightarrow{B}\| \cos\phi \tag{1}$$

where ϕ is the angle between the two vectors. That dipole placed in that magnetic field will feel a torque $\vec{\tau}_d$ that will try to align it with the magnetic field, thus minimizing its potential energy (Acosta, 2006). This torque is equivalent to

$$\vec{\tau}_d = \vec{\mu} \times \overrightarrow{B} = \|\vec{\mu}\|\|\overrightarrow{B}\| \sin\phi \tag{2}$$

Because $\sin\phi$ increases as ϕ increases, the greater the angle between the dipole and the magnetic field, the greater the torque felt by that dipole.

This same principle can be used with the magnetic fields. Assuming the magnetic fields of the driver gear's magnets are the magnetic field $\vec{B}$ from the preceding model and the magnets along the circumference of the driven wheel are the dipoles $\vec{\mu}$, then the greater the angle between the two gears, the greater the torque felt by the driven gear.

Calculating this difference in the displacements $\Delta\vec{\Theta}$ of the driver gear $\Delta\vec{\theta}_i$ (i for *input*) and the driven gear $\Delta\vec{\theta}_o$ (o for *output*) is straightforward:

$$\Delta\vec{\Theta} = \Delta\vec{\theta}_i - \Delta\vec{\theta}_o \tag{3}$$

I hypothesize that there exists some function $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ that yields that torque given $\Delta\vec{\Theta}$. Using this function, the acceleration of driven gear given the moment of inertia I_o of the load being driven can be determined using Newton's second law of motion:

$$\begin{gathered}\vec{\tau}_{net} = I\vec{\alpha} \quad \Rightarrow \quad \vec{\alpha} = \frac{\vec{\tau}_{net}}{I} \\ \vec{\alpha}_o = \frac{\vec{\mathrm{T}}(\Delta\vec{\Theta})}{I_o}\end{gathered} \tag{4}$$

Later, we will need to use the differential form of this equation to model the behaviour of the system. Recalling that acceleration is the second derivative of displacement and that the displacement of the driver gear accelerating at a constant rate $\vec{\alpha}_i$ for Δt s is $1/2\vec{\alpha}\Delta t^2$, Eq. (4) can also be expressed as[3]

$$\Delta\ddot{\vec{\theta}}_o = \frac{\vec{\mathrm{T}}\left(\frac{\vec{\alpha}_i\Delta t^2}{2} - \Delta\vec{\theta}_o\right)}{I_o} \tag{5}$$

1.4 Hypothesizing Slipping

As the driver gear first begins to accelerate, its angular displacement is momentarily greater than that of the driven gear. Assuming my hypothesis from Section 1.3 is correct, the driven gear now feels a torque (due to the difference in displacements) and begins to accelerate at a rate $\vec{\alpha}_o$ dictated by $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ which is not necessarily equal to $\vec{\alpha}_i$.

[3]For the sake of readability, Newton's notation for differentiation will be used throughout this essay, where $\Delta\dot{\vec{\theta}} = \vec{\omega} = \frac{d\Delta\vec{\theta}}{dt}$ and $\Delta\ddot{\vec{\theta}} = \vec{\alpha} = \frac{d^2\Delta\vec{\theta}}{dt^2}$. This notation specifically compares a variable (in this case, displacement) against time ("Newton's Notation for Differentiation," n.d.).

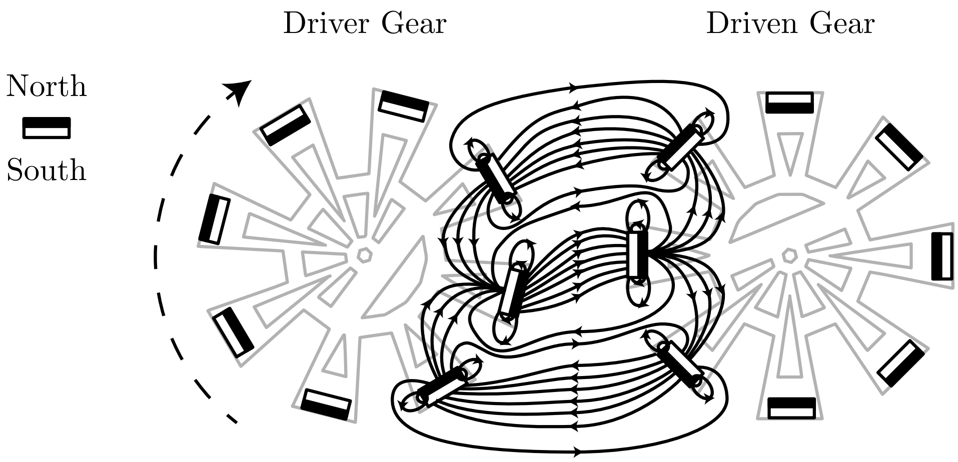


Figure 3: A simplified illustration of the magnetic fields between two misaligned magnetic gears when $\Delta\vec{\Theta} = 0.25\,\mathrm{rad}$ (a torque is trying to accelerate the driven gear).

Now, either one of two things happens. On the one hand, the driver gear could accelerate so quickly that it "escapes" the magnetic field of the driven gear and no longer accelerates it. On the other hand, the driven gear could momentarily accelerate at a rate *greater* than that of the driver gear in order to reduce the difference in their angular displacements, thus ensuring the gear continues to accelerate. However, if this were to happen, the driver gear would also have to eventually decelerate the driven gear because the driven gear cannot accelerate at a rate greater than the source of its torque indefinitely.

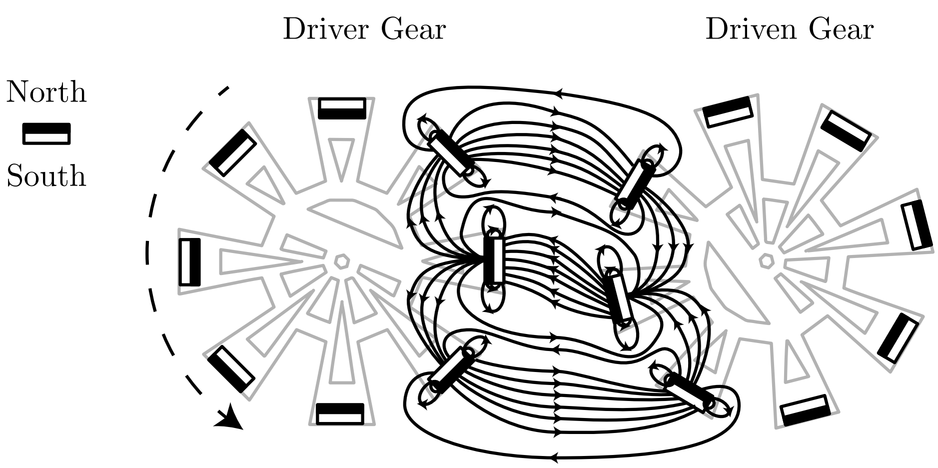


Figure 4: A simplified illustration of the magnetic fields between two misaligned magnetic gears when $\Delta\vec{\Theta} = -0.25\,\mathrm{rad}$ (a torque is trying to decelerate the driven gear).

Which of these outcomes occurs depends on just how fast the driver gear is being accelerated and on the maximum torque the driver gear can apply to the driven gear as dictated by the $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function of that particular gear train.

In order to understand this behaviour, the nature of $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ needs to be determined.

Before doing so definitively, we can predict some of the function's properties and use this prediction to validate our results. Namely, $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ is likely periodic. I say this because the function models the behaviour of two symmetric rotating gears. Specifically, because the gears each have eight magnets evenly distributed along their perimeter and the polarity of the outer face of each magnet alternates, there are likely four periods of $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ per full rotation of the gear (2π rad), leading to an expected period of $\pi/2$ rad. Assuming this is the case, then an experiment determining the nature of $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ does not need to look at a full rotation of the magnetic gear train, but only $\pi/2$ rad.

Furthermore, if $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ has a period of $\pi/2$ rad, then slipping likely occurs in the gear train when $\Delta\vec{\Theta}$ surpasses $\pi/2$ rad. This is the point of no return: once the displacement of the driver gear exceeds that of the driven gear by an entire period, slipping has already occurred and, because the driver gear is accelerating at a rate that the driven gear is clearly unable to keep up with, slipping will likely continue forever.

2 Derivation

This section summarizes the determination of the nature of $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ and discusses the moment of inertia I_o of the driven gear.

2.1 Generating the Torque Curve

The nature of $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ will be determined experimentally.

2.1.1 Variables

The variables of this experiment are summarized in Table 1 below.

Table 1: Variables in Torque Curve Generation

	Variable	**Value**
Independent	Angular displacement of the driver gear	
Dependent	Torque felt by the driven gear	
Controlled	Magnetic gears used	As designed in Section 1.2.1
	Displacement of driven gear	$0.00° \pm 0.25°$
	Separation of the gears	$0.045\,\mathrm{m} \pm 0.001\,\mathrm{m}$

Note that this separation of the gears ($0.045\,\mathrm{m} \pm 0.001\,\mathrm{m}$) is chosen because any closer and the pull of the magnets begins to bend the axles they are mounted on. Any farther and there would not be as significant a variation in the torque applied from the driver gear to the driven gear as a function of their displacements.

2.1.2 Apparatus

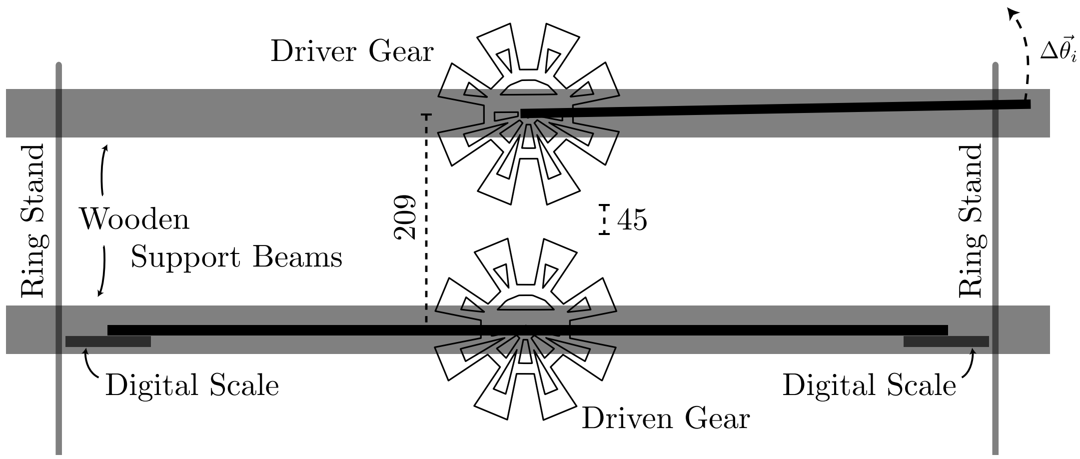


Figure 5: The apparatus used to produce the torque curve. All measurements are in mm $\pm$ 1 mm.

To generate the torque curve, the gears are positioned on top of each other.

The top gear is the driven gear, and a long arm extending from its centre is attached to it. This arm serves to control the angular displacement of the gear with stability. The displacement is measured with a Vernier rotary motion sensor (precise to 0.25°) attached to the end of the axle.

As for the driven gear, a long arm extending well beyond its diameter is mounted onto its axis and attached to the gear's frame such that the arm is parallel to the ground[4]. Digital scales are placed no more than 0.0010 m $\pm$ 0.0001 m under the ends of the arm (without touching them)[5]. A different Vernier rotary motion sensor is mounted to the end of the driven gear's axle to measure its displacement. As the driver gear is rotated, the torque it applies on the driven gear rotates the latter such that the one end of its arm pushes down on the scale, yielding a reading that is recorded[6]. After a certain displacement, the arm of the driven gear begins to push on the other scale. Because the arm that is fixed to the gear pushes against the scales, its angular displacement is negligible.

It is important that magnetic materials other than the magnets on the gears be present in the apparatus so as not to interfere in the torque transfer. As such, the beams suspending the gears were wooden and the arms attached to the gears were plastic.

The experiment is repeated a total of three times.

[4]This is verified with a level.
[5]This value is verified using precise calipers.
[6]The peculiar nature of this reading will be discussed briefly in Section 2.1.4 and in detail in Appendix B

2.1.3 Raw Data

In total, 177 data points were collected for this trial of the experiment. The minimum driver gear angular displacement, being $0.00° \pm 0.25°$, is the first data point that no reading was provided by the scales. The maximum driver gear angular displacement, being $83.00° \pm 0.25°$, is the point at which the arm attached to the driver gear to precisely control its displacement was almost vertical in the air and can no longer be rotated without disturbing the rest of the apparatus, potentially leading to systematic error in the data collected thereafter. Maintaining a perfectly constant increase in the displacement of the driver gear from one data point to the next was difficult. However, I sought to have an increment of about 0.25° between each data point as I approached where I suspected a peak would occur in the $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function so as to have sufficient data to precisely analyze, whereas an increment of about 0.75° during where I suspected the seemingly linear intervals of the curve would suffice. Three trials were performed to minimize random error.

A sample of the raw data collected in this trial is provided in Table 2.

Table 2: Sample Raw Data from Trial 1 of Torque Curve Generation

Data Point #	$\boldsymbol{\Delta\vec{\theta}_i}$ (° ± **0.25** °)	$\boldsymbol{\Delta\vec{\theta}_o}$ (° ± **0.25** °)	**Left Scale Reading** (g ± **0.01** g)	**Right Scale Reading** (g ± **0.01** g)
1	0.00	0.00	0.00	0.00
2	0.25	0.00	0.00	0.11
3	0.25	0.00	0.00	0.41
⋮	⋮	⋮	⋮	⋮
113	42.25	0.00	0.00	2.32
114	42.25	0.00	0.00	1.63
115	42.75	0.00	0.85	0.00
⋮	⋮	⋮	⋮	⋮
175	82.00	0.00	26.99	0.00
176	82.50	0.00	24.75	0.00
177	83.00	0.00	22.82	0.00

Furthermore, the arm attached to the driven gear has a total length of $0.824\,\mathrm{m} \pm 0.001\,\mathrm{m}$ and a width of $0.0552\,\mathrm{m} \pm 0.0005\,\mathrm{m}$.

The full raw datasets from each trial are available in Appendix I.

2.1.4 Processed Data

Next, the data needs to be processed. An example of each calculation performed is provided using data point 114 of Trial 1.

First, each angular measurement needs to be converted from degrees into radians, the Standard International unit for an angle. Then, the difference in angular displacements and its uncertainty can be found[7]:

$$\begin{aligned} \Delta\vec{\theta}_{i_{\text{rad}}} &= \Delta\vec{\theta}_{i_{\text{deg}}} \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 42.25° \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 0.737\,\text{rad} \end{aligned} \qquad \begin{aligned} U_{\Delta\vec{\theta}_{i_{\text{rad}}}} &= U_{\Delta\vec{\theta}_{i_{\text{deg}}}} \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 0.25° \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 0.004\,\text{rad} \end{aligned} \tag{6}$$

$$\begin{aligned} \Delta\vec{\theta}_{o_{\text{rad}}} &= -\Delta\vec{\theta}_{o_{\text{deg}}} \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 0.00° \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 0.000\,\text{rad} \end{aligned} \qquad \begin{aligned} U_{\Delta\vec{\theta}_{o_{\text{rad}}}} &= U_{\Delta\vec{\theta}_{o_{\text{deg}}}} \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 0.25° \cdot \frac{2\pi \text{ rad}}{360°} \\ &= 0.004\,\text{rad} \end{aligned} \tag{7}$$

$$\begin{aligned} \Delta\vec{\Theta} &= \Delta\vec{\theta}_{i_{\text{rad}}} - \Delta\vec{\theta}_{o_{\text{rad}}} \\ &= 0.737\,\text{rad} - 0.000\,\text{rad} \\ &= 0.737\,\text{rad} \end{aligned} \qquad \begin{aligned} U_{\Delta\vec{\Theta}} &= U_{\Delta\vec{\theta}_{i_{\text{rad}}}} + U_{\Delta\vec{\theta}_{o_{\text{rad}}}} \\ &= 0.004\,\text{rad} + 0.004\,\text{rad} \\ &= 0.008\,\text{rad} \end{aligned} \tag{8}$$

Recall that a torque $\vec{\tau}$ is a force $\vec{F}$ acting at a distance $\vec{r}$ from an axis of rotation:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{9}$$

The radius $\vec{r}$ at which the force is pushing on the scale needs to be calculated.

[7]In this essay, the variable U_n denotes the absolute uncertainty of the variable n. An uncertainty $U_{rel\,n}$ denotes the relative uncertainty of n expressed as a decimal. For example, an uncertainty of 1% has a relative uncertainty U_{rel} of 0.01.

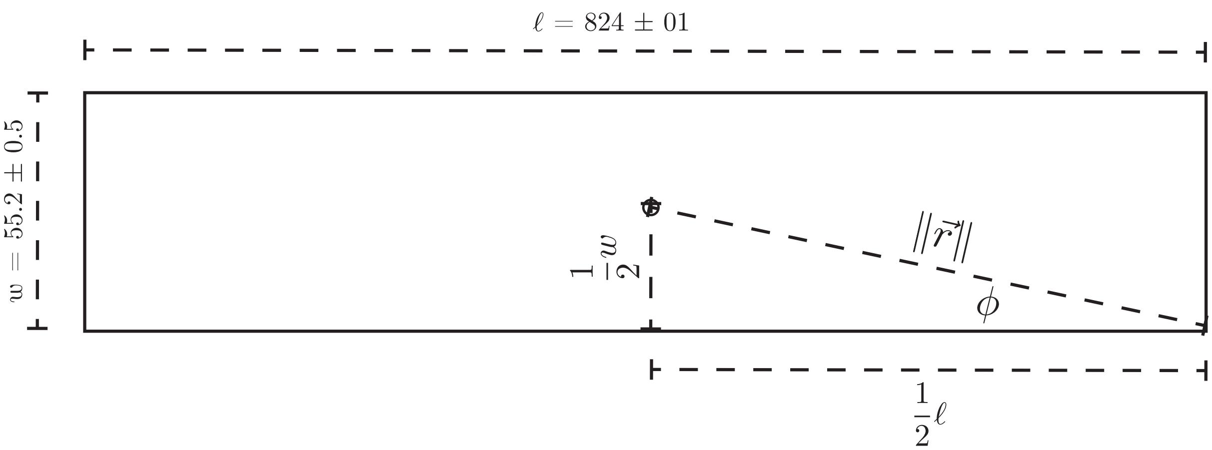
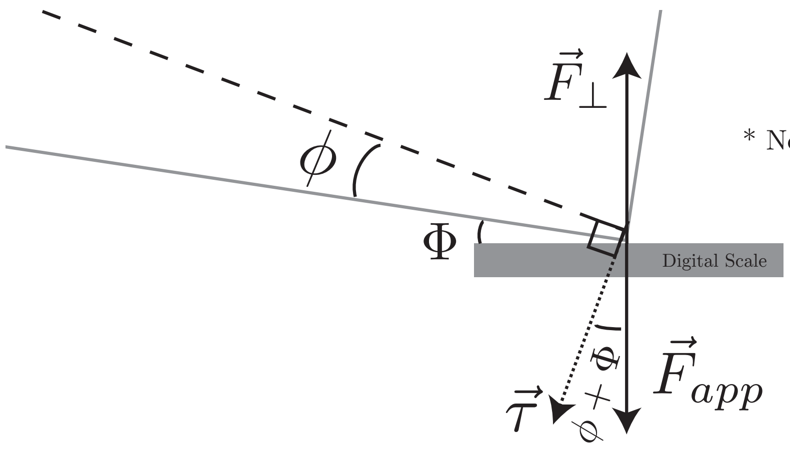


Figure 6: The geometry of the arm mounted to the driven gear (not to scale). All measurements are in mm ± mm

$$\begin{aligned} \|\vec{r}\| &= \frac{1}{2}\sqrt{w^2+\ell^2} \\ &= \frac{1}{2}\sqrt{(0.0552\,\mathrm{m})^2+(0.824\,\mathrm{m})^2} \\ &= 0.413\,\mathrm{m} \end{aligned} \qquad \begin{aligned} U_{rel\,\|\vec{r}\|} &= \frac{1}{2}\left(2U_{rel\,w}+2U_{rel\,\ell}\right) \\ &= \frac{0.0005\,\mathrm{m}}{0.0552\,\mathrm{m}}+\frac{0.001\,\mathrm{m}}{0.824\,\mathrm{m}} \\ &= 0.0103 \end{aligned} \tag{10}$$

$$\begin{aligned} \phi &= \arctan\left(\frac{\frac{1}{2}w}{\frac{1}{2}\ell}\right) \\ &= \arctan\left(\frac{0.0276\,\mathrm{m}}{0.412\,\mathrm{m}}\right) \\ &= 0.0669\,\mathrm{rad} \end{aligned} \qquad \begin{aligned} U_{rel\,\phi} &= U_{rel\,w}+U_{rel\,\ell} \\ &= \frac{0.0005\,\mathrm{m}}{0.0552\,\mathrm{m}}+\frac{0.001\,\mathrm{m}}{0.824\,\mathrm{m}} \\ &= 0.0103 \end{aligned}$$

Recall that the readings the scales were producing were not direct force readings themselves. In order to convert these readings of gram-force to the Standard International unit of Newtons, we multiply them by the gravitational constant $0.00981\,\mathrm{N\,g^{-1}}$ (Thompson & Taylor, 2008). If the driven arm is pushing on the right scale, the torque will be positive because the driver gear is accelerating the driven gear. If the driven arm is pushing on the left scale, the torque will be negative because the driver gear is decelerating the driven gear. Furthermore, note that gravity is negligible because, essentially, the arm attached to the driven gear was perpendicular to gravity. This is elaborated on in Appendix B.



* Note that Φ is greatly exaggerated.

Figure 7: A force diagram describing the observations at each end of the arm mounted to the driven gear where the arm meets the scale.

$$\begin{aligned} \|\vec{F}_{app}\| &= \|\vec{F}_\perp\| \\ \vec{\tau} &= \frac{\|\vec{r}\|\|\vec{F}_{app}\|}{\cos(\phi + \Phi)} \\ &= \frac{0.413\,\mathrm{m} \cdot 1.63\,\mathrm{g} \cdot 0.00981\,\mathrm{N\,g^{-1}}}{\cos(0.0669\,\mathrm{rad} + 0\,\mathrm{rad})} \\ &= 0.00662\,\mathrm{N\,m} \end{aligned} \tag{11}$$

$$\begin{aligned} U_{rel\,\vec{\tau}} &= U_{rel\,\|\vec{r}\|} + U_{rel\,\|\vec{F}\|} + U_{rel\,\phi} \\ &= 0.0103 + \frac{0.01\,\mathrm{g}}{1.63\,\mathrm{g}} + 0.0103 \\ &= 0.0267 \end{aligned} \qquad \begin{aligned} U_{\vec{\tau}} &= U_{rel\,\vec{\tau}} \cdot \vec{\tau} \\ &= 0.0267 \cdot 0.00662\,\mathrm{N\,m} \\ &= 0.0002 \end{aligned} \tag{12}$$

The final value for the torque applied is rounded to the maximum precision permitted by the absolute uncertainty ($0.0066\,\mathrm{N\,m} \pm 0.0002\,\mathrm{N\,m}$).

Table 3 provides a sample of the processed data from Trial 1. The complete processed datasets from all trials are available in Appendix J.

Table 3: Sample Processed Data from Trial 1 of Torque Curve Generation

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	**Torque** (N m)	±	**Torque Unc.** (N m)
1	0.000	0	±	0
2	0.003	0.00045	±	0.00005
3	0.003	0.00166	±	0.00007
⋮	⋮	⋮	⋮	⋮
113	0.737	0.0094	±	0.0002
114	0.737	0.0066	±	0.0002
115	0.747	-0.00345	±	0.00009
⋮	⋮	⋮	⋮	⋮
175	1.431	-0.110	±	0.002
176	1.440	-0.100	±	0.002
177	1.449	-0.093	±	0.001

When the resulting torques are plotted against their respective differences in angular displacements, the nature of the $\vec{T}(\Delta\vec{\Theta})$ function is revealed.

Figure 8: Torque Curves for all Trials

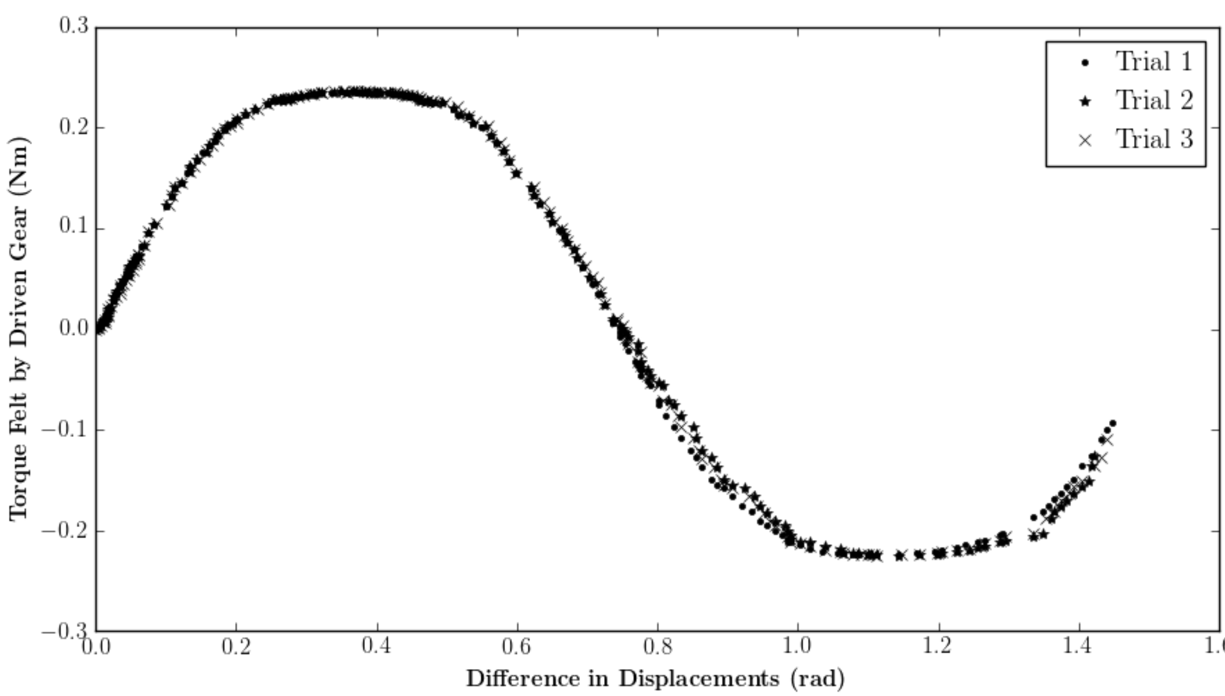
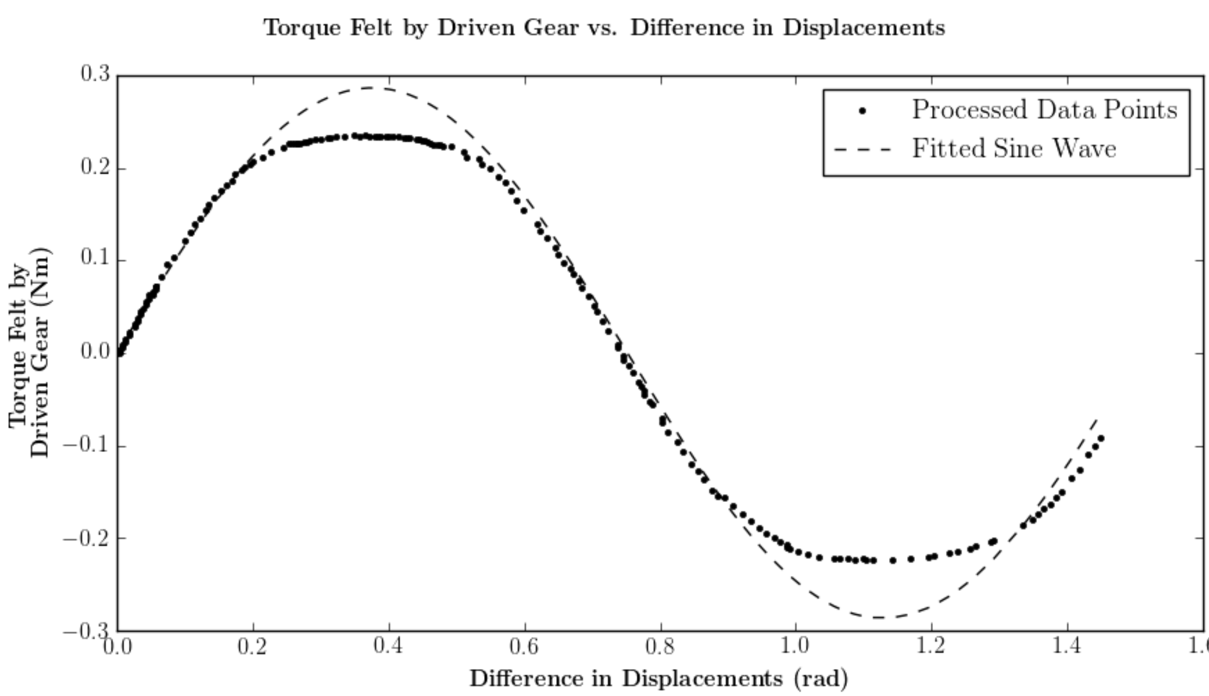


Figure 8 demonstrates the consistency of the collected data across all three trials, suggesting a high degree of precision[8].

2.1.5 Data Interpolation

Even though we often try to generalize datasets with a mathematical trend, we will not do that here. At first glance, the data appears to be sinusoidal: the curve passes through the origin (0 N m of torque was produced for a difference in displacements of 0 rad) and appears to oscillate. Figure 9 illustrates an attempt at fitting a sine wave to the data using a program I wrote available in Appendix E. However, as can be seen, the data is not quite sinusoidal, with flatter peaks than expected. Furthermore, we cannot be sure that a torque curve generated under different conditions for a different gear train (e.g., gears closer together or farther apart, driver gear and driven gear with different numbers of magnets, stronger or weaker magnets) would follow this same trend.

Figure 9: Torque Curve for Trial 1 with Attempted Sine Wave Fit



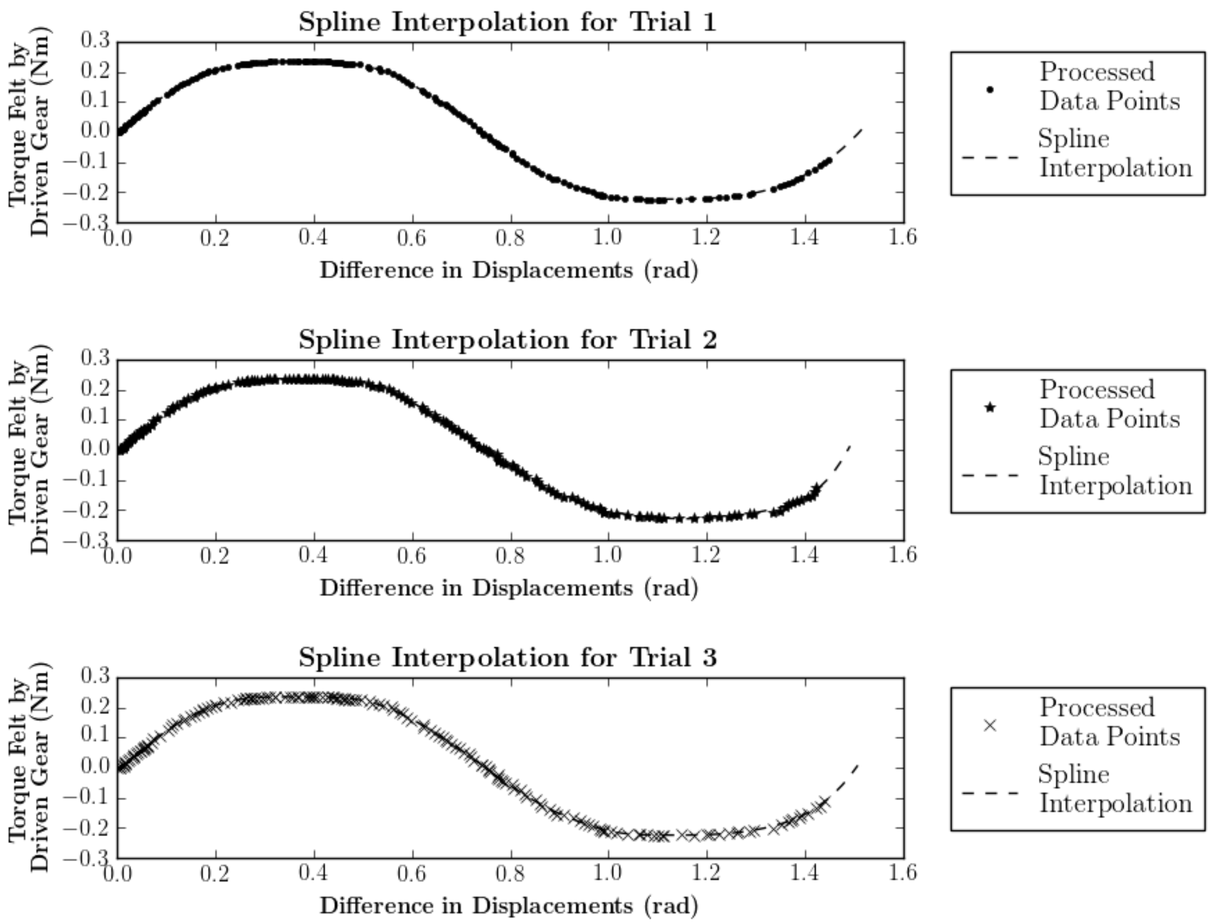
For these reasons, we are not going to fit a specific type of mathematical function to the data. Yet, we still need to know the magnitude of the generated torque for any difference in angular displacements, even if the desired value is not one I explicitly measured. So, we will interpolate the data. Taking this approach means that any torque curve can be used to

[8]Note that while the error bars of each point are present on the plot, they are too small to be rendered.

determine acceleration regardless of the nature of the curve and the ease of precisely fitting a function to said data.

To accomplish this, I wrote a Python program based on the open-source SciPy.Interpolate library that comes up with a spline interpolation[9] for each trial. The program can be found in Appendix F, and the spline interpolations it fit to the data from each trial can be seen in Figure 10.

Figure 10: Torque Curves with Spline Interpolation Fits for All Trials



The fact that the spline interpolations closely follow the data points for each trial shows that they are precise fits. While the error associated with a fit would normally be determined using something like maximum and minimum slopes found by considering the uncertainty of each point used to produce said fit, this is not no straightforward with spline interpolations. The torque felt by the driven gear given the difference in the displacements of the driver and

[9]Spline interpolation essentially fits polynomials to each pair of points in a dataset while taking the proximity of other points into account so as to produce a smooth, precise curve (n.d.). Spline interpolations will not necessarily pass through all the points in the dataset that produced them.

driven gears can be determined by averaging the three individual torque values outputted by each spline interpolation at the given difference in displacements, and the uncertainty of this value can be determined by taking a range uncertainty.

The period of the processed datasets as shown in Figure 8 appears to be approximately 1.5 rad, which is close to the expected value of $\pi/2$ rad (approximately 1.57 rad), representing one quarter rotation of the driver wheel. This suggests a certain degree of accuracy in the dataset.

In order to definitively calculate the period of each spline interpolation, each wave was truncated once it reached $\vec{\tau} = 0\,\text{rad}$ for the third time: the first time $\vec{\tau}$ is $0\,\text{rad}$ is at the origin, the second time is halfway through the period, and the third time is at the end of the period. The $\Delta\vec{\Theta}$ value for each of these points is given in Table 4, which also provides the average period[10] of the entire dataset.

Table 4: Periods of Spline Interpolations

Trial #	**Period (rad ± rad)**
1	1.51
2	1.49
3	1.50
Average	1.50 ± 0.01

$$\%_{accuracy} = \frac{1.50\,\text{rad}}{\pi/2\,\text{rad}} \approx 96\% \tag{13}$$

The average period of the dataset does fall does slightly short of the expected value of $\pi/2$rad. However, at 96% of its expected value, this error is insignificant and suggests a fairly accurate result.

This error could likely have been reduced by collecting data over one full period of the curve. However, the apparatus did not permit a precise measurement of driver angular displacements of beyond $\sim 83°$. This could have been improved in the experimental setup.

2.2 Moment of Inertia & Resistive Torques

The moment of inertia I_o of the driven gear also needs to be known to calculate acceleration. However, in the case of this exploration, sustainability can be predicted for any arbitrary moment of inertia. However, I did determine the moment of inertia of my driven gear mounted on its axle out of curiosity in Appendix C.

[10] The uncertainties of the individual periods for each trial could not accurately be determined but are irrelevant as a range uncertainty is used to find the average period of the datasets.

Given a restrictive word limit, the main body of this essay will focus on accelerating magnetic gear trains with constant moments of inertia. However, analyses considering moments of inertia that vary with velocity (as is the case when considering air resistance) and decelerating systems are available in Appendix H.

3 Implementation

Recall from Eq. (5) that

$$\Delta\ddot{\vec{\theta}}_o = \frac{\vec{\mathrm{T}}\left(\frac{\vec{\alpha}_i \Delta t^2}{2} - \Delta\vec{\theta}_o\right)}{I_o}$$

and that the $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function is modelled by three spline interpolations. This differential equation cannot be solved[11]. To approximate it, numerical methods must be used.

3.1 Using the Classical Runge-Kutta Method

The numerical method we will use is the classical Runge-Kutta method[12].

The classical Runge-Kutta method requires starting conditions for the system as well as an increment h for the independent variable (being time elapsed). The error of the method is proportional to h^5; as h decreases, we get significantly more accurate results at the expense of more computations to perform. I wrote a program to perform these calculations; it is available in Appendix G.

The Runge-Kutta methods are only able to approximate first-order differential equations. However, our acceleration function is a second-order equation. Therefore, it needs to be converted to a system of first-order equations. To do this, we begin by substituting $\vec{\omega}_o$ for $\Delta\dot{\vec{\theta}}$:

$$\dot{\vec{\omega}}_o = \frac{\vec{\mathrm{T}}\left(\frac{\vec{\alpha}_i \Delta t^2}{2} - \Delta\vec{\theta}_o\right)}{I_o} \tag{14}$$

[11] This is the case with many differential equations.

[12] The Runge-Kutta methods are a group of numerical methods for approximating differential equations by taking a weighted average slope of several preceding approximated points to estimate the next one (Derrick & Grossman, 1981; O'Neil, 1983). While there are various techniques within the broader scope of the Runge-Kutta methods, the "classical Runge-Kutta method" is the most common.

It therefore follows that[13]:

$$\vec{\omega}_o\left(\Delta t\right) = \frac{d\Delta\vec{\theta}_o}{d\Delta t} \quad \Rightarrow \quad d\Delta\vec{\theta}_o = \vec{\omega}_o\left(\Delta t\right)\, d\Delta t$$

$$\begin{aligned}\dot{\vec{\omega}}_o = \frac{d\vec{\omega}_o}{d\Delta t} &= \vec{\alpha}_o\left(\Delta t, \Delta\vec{\theta}_o\left(\Delta t\right), \vec{\omega}_o\left(\Delta t\right)\right) \\ d\vec{\omega}_o = \dot{\vec{\omega}}_o\, d\Delta t &= \vec{\alpha}_o\left(\Delta t, \Delta\vec{\theta}_o\left(\Delta t\right), \vec{\omega}_o\left(\Delta t\right)\right) d\Delta t \\ \Delta\vec{\theta}_o\left(\Delta t + h\right) &= \Delta\vec{\theta}_o\left(\Delta t\right) + d\Delta\vec{\theta}_o \\ \vec{\omega}_o\left(\Delta t + h\right) &= \vec{\omega}_o\left(\Delta t\right) + d\vec{\omega}_o \\ \Delta\ddot{\vec{\theta}}_{o\,\text{next}} &= \vec{\alpha}\left(\Delta t + h, \Delta\vec{\theta}_o\left(\Delta t + h\right), \vec{\omega}_o\left(\Delta t + h\right)\right)\end{aligned} \tag{15}$$

For our system, we know that at $\Delta t = 0\,\text{s}$, the driven gear is not and has not been rotating. These are sufficient initial conditions.

$$\begin{aligned}\Delta\vec{\theta}_o\left(0\,\text{s}\right) &= 0\,\text{rad} \\ \vec{\omega}_o\left(0\,\text{s}\right) &= 0\,\text{rad}\,\text{s}^{-1}\end{aligned} \tag{16}$$

The classical Runge-Kutta method says that

$$\begin{aligned}d\Delta\vec{\theta}_o &= \frac{1}{6}\left(\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4\right) \\ d\vec{\omega}_o &= \frac{1}{6}\left(\mu_1 + 2\mu_2 + 2\mu_3 + \mu_4\right)\end{aligned} \tag{17}$$

where

$$\begin{aligned}\lambda_1 &= h\vec{\omega}_o\left(\Delta t\right) & \mu_1 &= h\vec{\alpha}_o\left(\Delta t, \Delta\vec{\theta}_o\left(\Delta t\right), \vec{\omega}_o\left(\Delta t\right)\right) \\ \lambda_2 &= h\left(\vec{\omega}_o\left(\Delta t\right) + \frac{\mu_1}{2}\right) & \mu_2 &= h\vec{\alpha}_o\left(\Delta t + \frac{h}{2}, \Delta\vec{\theta}_o\left(\Delta t\right) + \frac{\lambda_1}{2}, \vec{\omega}_o\left(\Delta t\right) + \frac{\mu_1}{2}\right) \\ \lambda_3 &= h\left(\vec{\omega}_o\left(\Delta t\right) + \frac{\mu_2}{2}\right) & \mu_3 &= h\vec{\alpha}_o\left(\Delta t + \frac{h}{2}, \Delta\vec{\theta}_o\left(\Delta t\right) + \frac{\lambda_2}{2}, \vec{\omega}_o\left(\Delta t\right) + \frac{\mu_2}{2}\right) \\ \lambda_4 &= h\left(\vec{\omega}_o\left(\Delta t\right) + \mu_3\right) & \mu_4 &= h\vec{\alpha}_o\left(\Delta t + \frac{h}{2}, \Delta\vec{\theta}_o\left(\Delta t\right) + \lambda_3, \vec{\omega}_o\left(\Delta t\right) + \mu_3\right)\end{aligned} \tag{18}$$

By performing these computations thousands of times, we can model the driven gear's displacement, velocity, and acceleration over time.

[13] Note that $\Delta\vec{\theta}_o\left(\Delta t\right)$ and $\vec{\omega}_o\left(\Delta t\right)$ refer to the driven displacement $\Delta\vec{\theta}_o$ and the driven velocity $\vec{\omega}_o$ at time Δt, respectively, and not the product of these values and time elapsed.

3.1.1 Avoiding Significant Error

For as useful as the Runge-Kutta method is in approximating our differential equation, there is a catch with how it pertains to our needs. The period of our torque wave is $1.50\,\text{rad} \pm 0.01\,\text{rad}$. If either gear displaces more than $1.50\,\text{rad} \pm 0.01\,\text{rad}$ in a single time interval of h s, the Runge-Kutta algorithm errs: it neglects the torque of the driver gear on the driven gear that was applied between that time interval of $[\Delta t, \Delta t + h[$.

For this reason, a fail-safe needs to be built into my Runge-Kutta program to detect if this error occurred. The program halts if the driver gear exceeds a displacement of $\pi/24$ rad in any single time interval. This value guarantees almost 12 time intervals per period of the torque wave—enough measurements to ensure accuracy without being too restrictive in the algorithm's operation.

3.2 Error Propagation with the Runge-Kutta Method

Because the error associated with the classical Runge-Kutta method is so insignificant compared to the uncertainty of the $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function, it will be ignored.

On the other hand, the provided driver gear acceleration and the moment of inertia of the driven gear have uncertainties that need to be considered.

Recall from Eq. (5) that

$$\Delta\ddot{\vec{\theta}}_o = \frac{\vec{\mathrm{T}}\left(\frac{\vec{\alpha}_i \Delta t^2}{2} - \Delta\vec{\theta}_o\right)}{I_o}$$

Let $U_{\vec{\mathrm{T}}(\Delta\vec{\Theta})}$ be the range uncertainty of the yielded torques from the spline interpolations:

$$U_{\vec{\mathrm{T}}(\Delta\vec{\Theta})} = \frac{\vec{\mathrm{T}}(\Delta\vec{\Theta})_{max} - \vec{\mathrm{T}}(\Delta\vec{\Theta})_{min}}{2} \tag{19}$$

Therefore, the uncertainty associated with the driven acceleration is calculated as follows:

$$U_{\Delta\ddot{\vec{\theta}}_o} = U_{\vec{\mathrm{T}}(\Delta\vec{\Theta})} + \Delta\ddot{\vec{\theta}}_o \left(\frac{U_{\vec{\alpha}_i}}{\vec{\alpha}_i} + \frac{U_{I_o}}{I_o}\right) \tag{20}$$

The uncertainty of the velocity and displacement of the driven gear at a given moment are calculated by treating the uncertainty of each value as the value itself:

$$\begin{aligned} \Delta\dot{\vec{\theta}}_o = \vec{\alpha}_o \Delta t \quad &\Rightarrow \quad U_{\Delta\dot{\vec{\theta}}_o} = U_{\vec{\alpha}_o} h \\ \Delta\vec{\theta}_o = \Delta\dot{\vec{\theta}}_o \Delta t + \frac{1}{2}\vec{\alpha}_o \Delta t^2 \quad &\Rightarrow \quad U_{\Delta\vec{\theta}_o} = U_{\Delta\dot{\vec{\theta}}_o} h + \frac{1}{2} U_{\vec{\alpha}_o} h^2 \end{aligned} \tag{21}$$

If slipping is deemed to begin when $\Delta\overrightarrow{\Theta}$ exceeds the period of the torque curve[14], being $1.50\,\mathrm{rad} \pm 0.01\,\mathrm{rad}$, then slipping could theoretically begin as early as the first measurement of $\Delta\overrightarrow{\Theta}$ plus its uncertainty $U_{\Delta\overrightarrow{\Theta}}$ exceeds $1.49\,\mathrm{rad}$. Similarly, slipping could theoretically begin as late as the last value of $\Delta\overrightarrow{\Theta}$ minus its uncertainty $U_{\Delta\overrightarrow{\Theta}}$ that is less than $1.51\,\mathrm{rad}$. The Runge-Kutta program will output this interval during which slipping could begin.

[14] Because the period of the torque curve we generated is $1.50\,\mathrm{rad} \pm 0.01\,\mathrm{rad}$, slipping will be detected by determining when the difference in displacements between the two gears reaches $1.50\,\mathrm{rad} \pm 0.01\,\mathrm{rad}$ instead of the expected $\pi/2\,\mathrm{rad}$.

4 Evaluation

Finally, we are ready to predict the sustainability of an accelerating magnetic gear train.

4.1 Predicting Sustainability Using the Runge-Kutta Program

The Runge-Kutta program produces four graphs: an acceleration-time graph, a velocity-time graph, a graph looking at the difference in displacements between the two gears over time, and displacement-time graph. Analyzing these graphs will reveal the behaviour of the system as it accelerates.

Consider a magnetic gear train with the $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function determined in Section 2.1 and a constant moment of inertia of $0.00257\,\mathrm{kg\,m^2} \pm 0.00006\,\mathrm{kg\,m^2}$ as found in Appendix C (although for our purposes, this value could be any arbitrary constant). This scenario replicates a magnetic gear train accelerating in a frictionless vacuum.

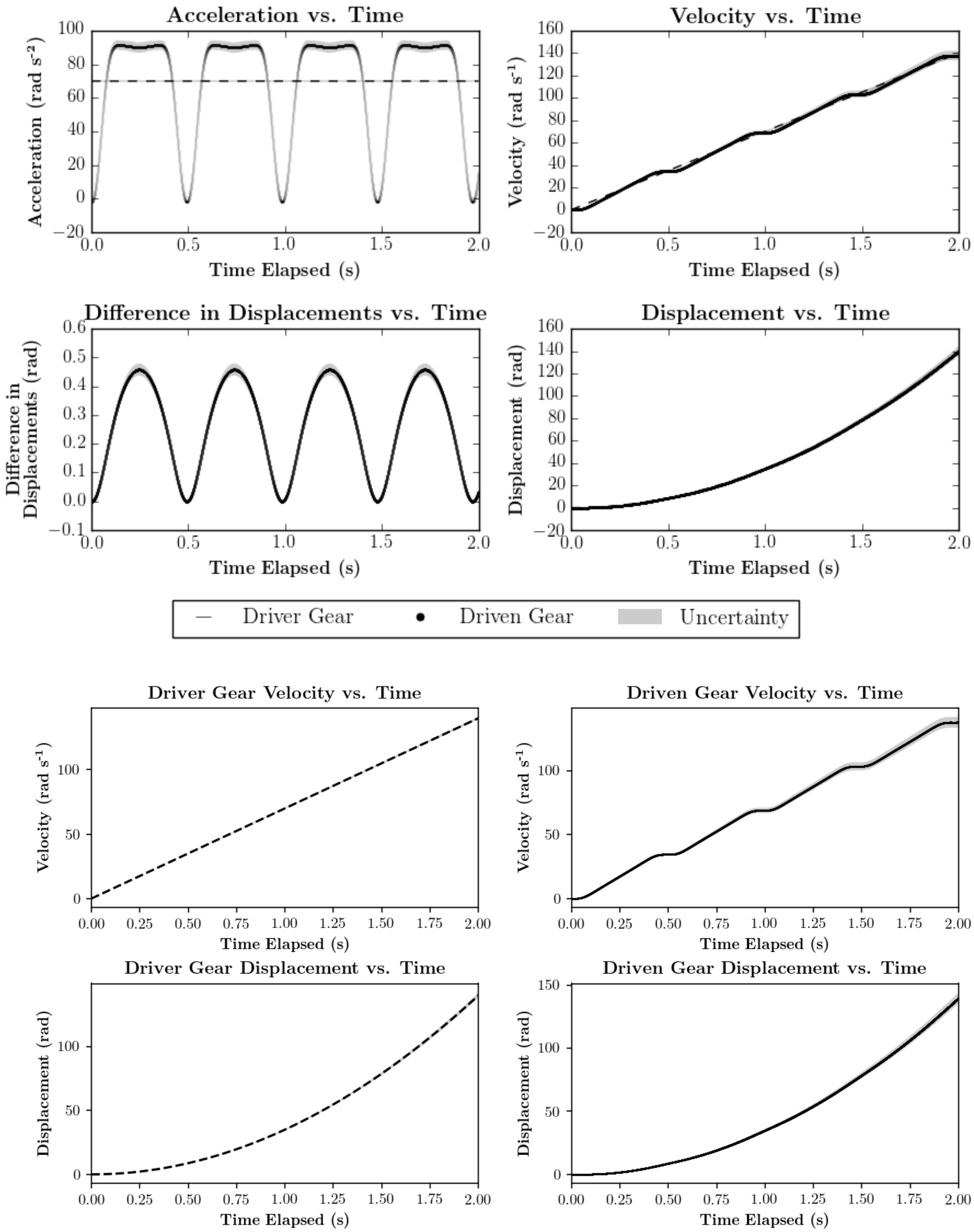
To begin, we will look at the behaviour of the system for the first two seconds of acceleration when the driver acceleration $\vec{\alpha}_i$ is $70.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$. The output of the Runge-Kutta program is shown in Figure 11[15,16,17].

[15]The command used to generate these plots is `python runge-kutta.py 70.0+-0.5 2 0.0001 0.00257+-0.00006 0+-0 0+-0`.

[16]Uncertainty on each plot is represented by the lightly shaded area surrounding each curve. It may be too small to be rendered given the hypothetical uncertainty values I have used.

[17]Additional graphs of the gears' velocities and displacements in isolation are provided for clarity.

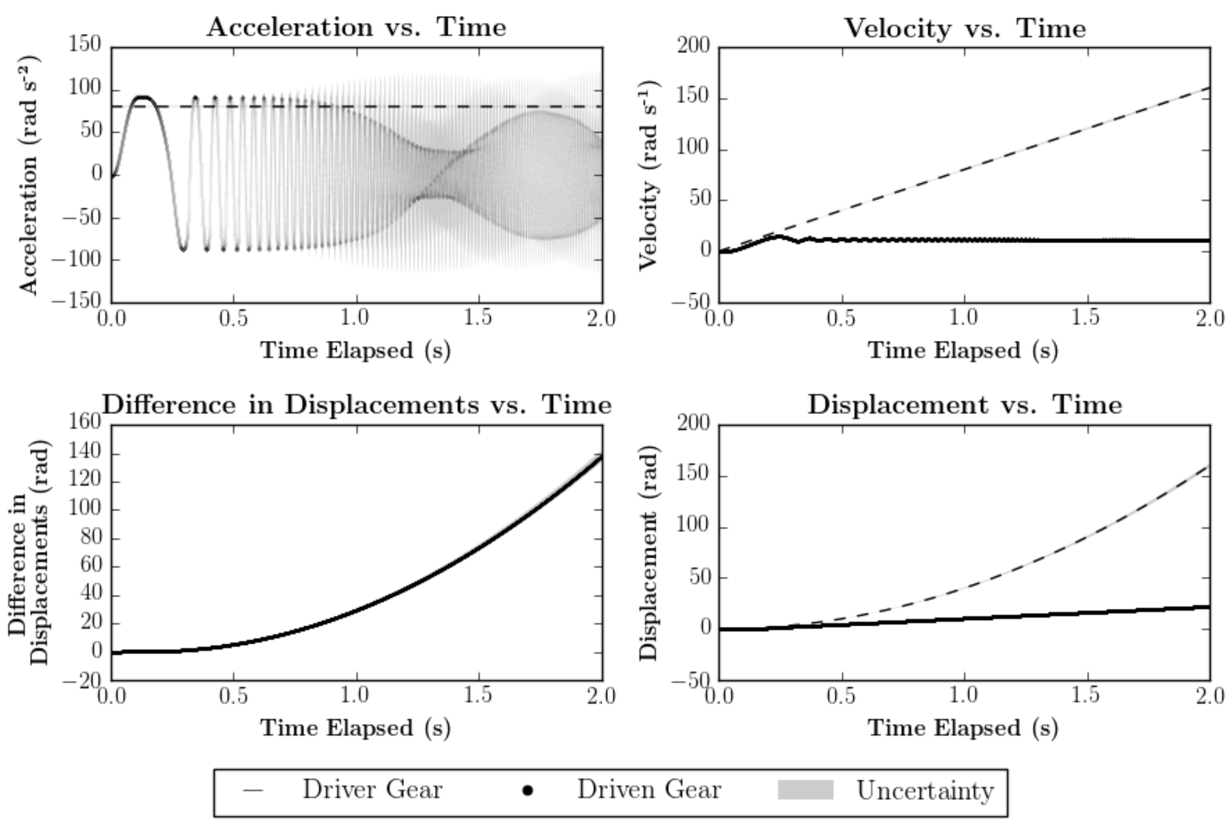
Figure 11: Runge-Kutta Program Output Given $\vec{\alpha}_i = 70.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ and $h = 0.0001\,\mathrm{s}$ for First $2\,\mathrm{s}$ of Acceleration



Already, these graphs reveal a lot about the behaviour of the driven gear while the driver gear is accelerating. In the acceleration-time plot, the acceleration of the driven gear does indeed regularly surpass that of the driver gear to minimize the difference in their displacements, thereby avoid slipping and demonstrating that my hypothesis from Section 1.4 is at least partially correct. In fact, the difference in displacements never surpasses the predicted critical value of 1.50 rad ± 0.01 rad, suggesting that slipping is not happening in this time frame. Furthermore, the velocity of the driven gear oscillates around that of the drive gear, showing that they are, on average, accelerating at the same rate; no slipping is occurring.

Next, we will try to induce slipping in this same system. Figures 12[18] and 13[19] show the output of the Runge-Kutta program when $\vec{\alpha}_i$ is $80.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ for the first two seconds and half second of acceleration, respectively.

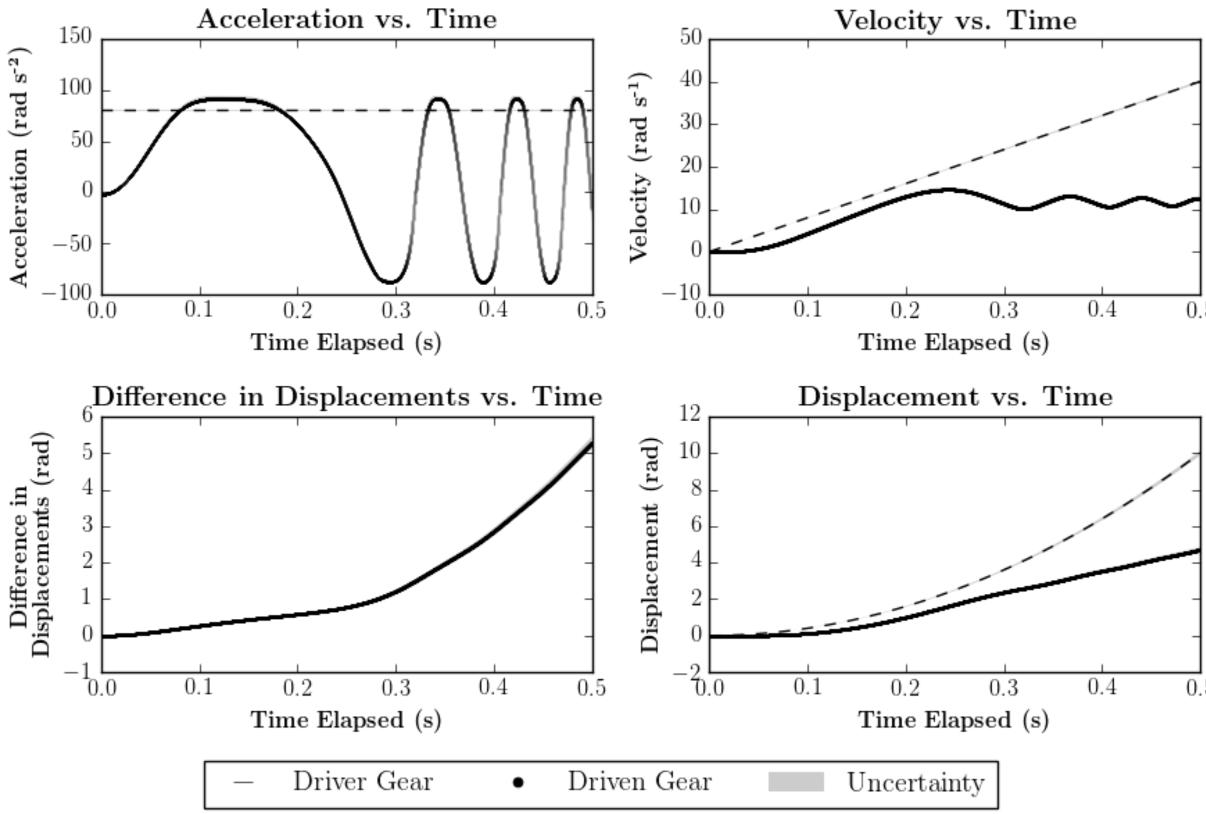
Figure 12: Runge-Kutta Program Output Given $\vec{\alpha}_i = 80.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ and $h = 0.0001\,\mathrm{s}$ for First 2 s of Acceleration



[18]The command used to generate these plots is `python runge-kutta.py 80.0+-0.5 2 0.0001 0.00257+-0.00006 0+-0 0+-0`.

[19]The command used to generate these plots is `python runge-kutta.py 80.0+-0.5 0.5 0.00001 0.00257+-0.00006 0+-0 0+-0`.

Figure 13: Runge-Kutta Program Output Given $\vec{\alpha}_i = 80.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ and $h = 0.00001\,\mathrm{s}$ for First $0.5\,\mathrm{s}$ of Acceleration



It is clear that slipping is occurring: in the displacement-time plot, the driven gear stops displacing at the same rate as the driven gear. Furthermore, the difference in the displacements of the gears never reaches $0\,\mathrm{rad}$ again after beginning to accelerate, and the critical value of $1.50\,\mathrm{rad} \pm 0.01\,\mathrm{rad}$ is surpassed after $0.320\,\mathrm{s} \pm 0.004\,\mathrm{s}$ of acceleration[20]. The acceleration-time plot interestingly shows the driven gear trying to catch up to the driver gear by momentarily accelerating at a greater rate than it, but then the driver gear "escapes" the pull of the driven gear. After this happens, the acceleration of the driven gear oscillates around $0\,\mathrm{rad\,s^{-2}}$. By consequence, the velocity of the driven gear stops matching that of the driver gear. This demonstrates that my hypothesis from Section 1.4 is correct.

4.2 Assumptions & Limitations

It goes without saying that analyzing an accelerating system for the first two seconds of its acceleration is not sufficient *on its own* to determine the sustainability of this acceleration. However, in conjunction with an understanding of the underlying physics dictating the system's behaviour, doing so can validate our hypotheses.

[20]This value is provided by the Runge-Kutta program.

The $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function referred to throughout this essay was produced from still measurements. It is being assumed that the nature of this function persists when it describes the relationship of two rotating objects. However, this is not unreasonable as magnetic fields do move at the speed of light.

5 Conclusions

5.1 Further Work

There is still a lot that is left to be explored in the realm of magnetic gears.

For example, how does the predictive model I have developed made come into play when there are more than two gears in the magnetic gear train? Do the magnetic fields of nearby gears that are neither the driver nor the driven gear of an accelerating pair interfere in the interaction of the latter two?

5.2 Final Remarks

The research question I set out to answer was *how long can a magnetic gear train can be accelerated at a given rate before slipping occurs, given the moment of inertia of the driven load?*

Ultimately, Figures 11 and 12 from Section 4 reveal that there is some value of $\vec{\alpha}_i$ between $70\,\mathrm{rad\,s^{-2}}$ and $80\,\mathrm{rad\,s^{-2}}$ that represents the maximum rate $\vec{\alpha}_{max}$ at which the magnetic gear train with these properties can be accelerated.

In this essay, I have discussed why and demonstrated that any accelerating magnetic gear train in a frictionless vacuum will accelerate *forever* provided it does so at a rate $\vec{\alpha}_i$ less than the maximum rate $\vec{\alpha}_{max}$ determined using a Runge-Kutta approximation. Otherwise, the gear begins slipping practically immediately and indefinitely.

Word Count: 3998

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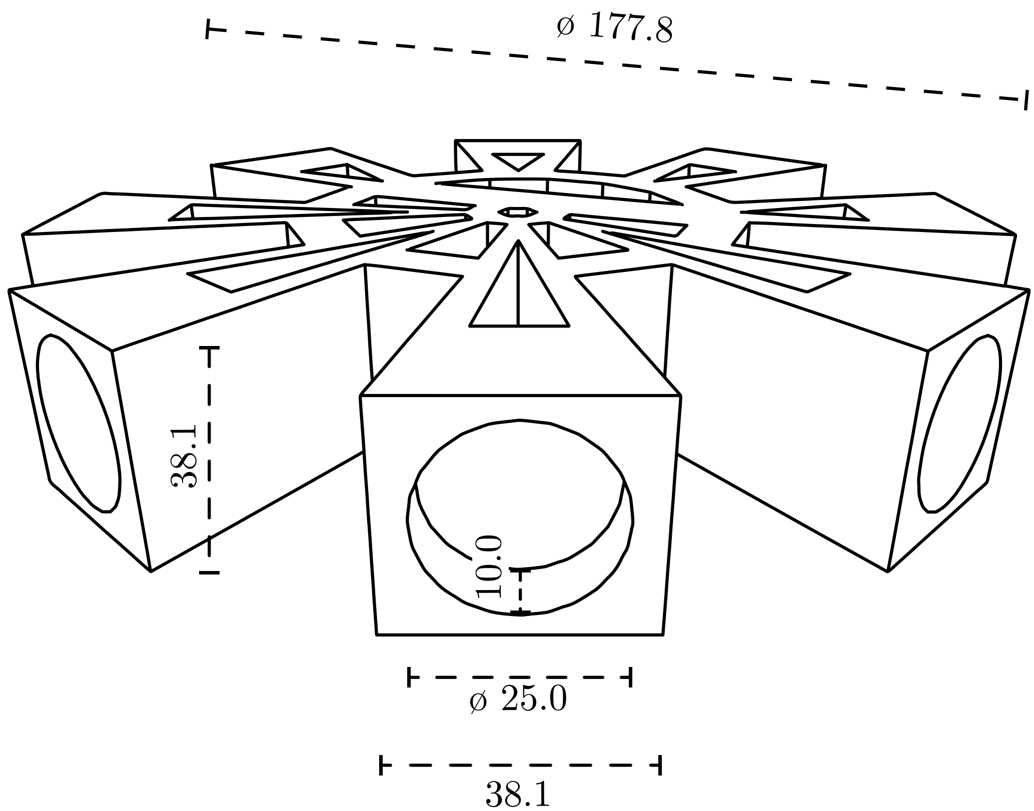
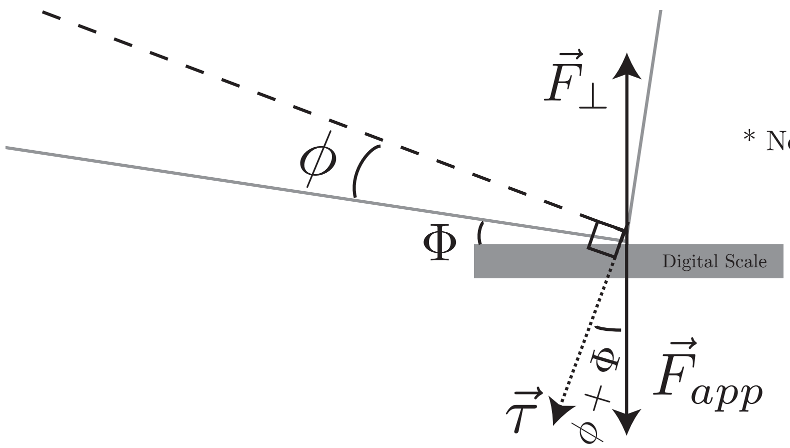
A Design of Magnetic Gear Frame



I designed the frame of my magnetic gears in Google Sketchup. It is in the shape of a hexadecagon (a 16-sided polygon). On every other edge of the polygon, a cylinder with the same dimensions as the magnets purchased was hollowed out so as to create a slot for them to slide into. Otherwise hollow sections of the frame (i.e., where no magnets were inserted) were trimmed away to save material and to reduce printing time. The magnets were inserted into their slots and epoxied into place as shown in Figure 2.

B Insignificance of Gravity in Torque Curve Generation



* Note that Φ is greatly exaggerated.

This is the same diagram as shown in Figure 7.

Because the arm attached to the driven gear did have the ability to rotate slightly, there is an additional angle Φ between the arm and the surface of the scale that needs to be considered. This excess rotation is necessary because when the arm attached to the driven gear pushed on the scale, it needs to do so from its corner. Should the arm have made contact parallel to the scale, it would have done so over a range of the arm's length, making it difficult to define the exact radius at which the force was being applied. The presence of the angle Φ between the arm and the ground means that gravity could theoretically have also applied a torque on the arm that needs to be considered as well. However, in reality, Φ is demonstrated to be $0.000\,\text{rad} \pm 0.004\,\text{rad}$ by the Vernier sensor, demonstrating that the torque applied by gravity is negligible. Note that the relative uncertainty of Φ cannot be determined for a measurement of $0.000\,\text{rad}$. However, the repeatability of this value across each data point is sufficient evidence to ignore this error as it is insignificant.

C Determining the Moment of Inertia of the Driven Gear

Recall that the moment of inertia I_o of the driven gear needs to be known in order to determine the acceleration of the gear given the torque that acts on it. While the exact value of I_o for the driven gear I designed was not necessary to explore the sustainability of the acceleration of any driven gear, I was still curious as to the approximate order of magnitude of a moment of inertia for an object with roughly the same mass and shape.

There are two ways the moment of inertia of my driven gear could be determined: by calculating it with calculus or by experimentation.

Moment of inertia can be defined as the distribution of mass in an object with respect to the distance r from the axis around which the object rotates:

$$I = \int r^2 dm \tag{22}$$

Given a complex shape like the frame of the driven gear with an uneven distribution of mass, this becomes very complicated very quickly. The moment of inertia could be approximated using known moments of inertia for simpler shapes, but even then, this would only be an approximation.

Alternatively, the moment of inertia could be determined experimentally using a bifilar pendulum (French, 2016). A bifilar pendulum is a pendulum where the mass m being oscillated is suspended by two lines of length L separated by a distance b. Essentially, when the suspended mass is twisted and released, the period T of its oscillations can be used to estimate its moment of inertia I using the following equation:

$$I = \frac{m\vec{g}b^2T^2}{4\pi^2 L} \tag{23}$$

where $\vec{g}$ is the acceleration due to gravity.

C.1 Variables

For this experiment, the independent variable is the moment of inertia of the gear. The dependent variable is the period of the oscillations of the gear. The controlled variables are the gear attached to the bifilar pendulum, the string used to suspend the gear (sewing thread), the length of the string ($0.562\,\mathrm{m} \pm 0.001\,\mathrm{m}$), and the separation of the strings ($0.0762\,\mathrm{m} \pm 0.0005\,\mathrm{m}$).

C.2 Apparatus

The driven gear, mounted on its axle, is attached to two lengths of sewing thread of negligible mass and suspended as shown in Figure 14.

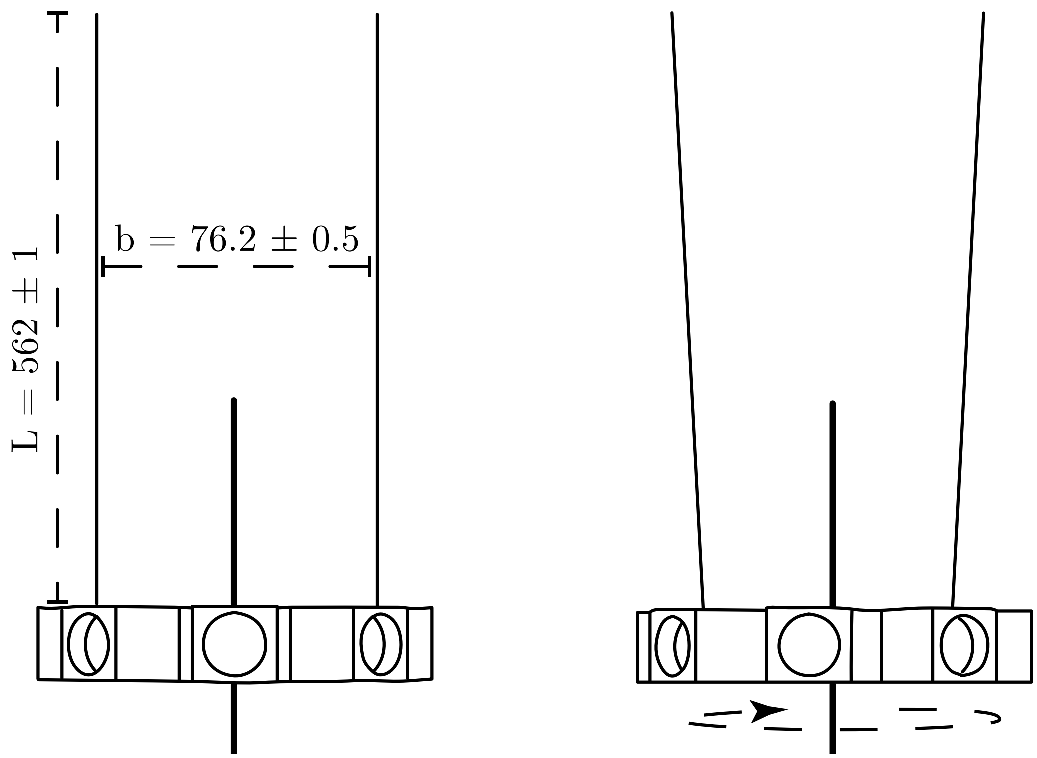


Figure 14: The setup of the bifilar pendulum. All measurements are in mm ± mm.

The gear is twisted and then released such that its translational motion is negligible (only rotational motion should be observed). After one oscillation, a timer is started and the mass is allowed to oscillate ten more times. After the ten oscillations, the timer is stopped. This number of allowed oscillations was chosen so as to minimize measurement uncertainty without giving too much time for the gear to begin to decelerate due to air resistance.

C.3 Raw Data

Table 5: Times for Ten Oscillations of Driven Gear in Bifilar Pendulum

Data Point #	**Time for 10 Oscillations (s $\pm$ 0.01 s)**
1	26.93
2	26.90
3	26.96
4	26.97
5	27.00

The length of each strand of sewing thread from which the gear is suspended is $0.562\,\mathrm{m} \pm 0.001\,\mathrm{m}$. The separation between the two threads is consistent along the entire lengths of the threads, at $0.0762\,\mathrm{m} \pm 0.0005\,\mathrm{m}$. The combined mass of the threads is $0.00\,\mathrm{g} \pm 0.01\,\mathrm{g}$. The mass of the gear and its axle is $0.55110\,\mathrm{kg} \pm 0.00005\,\mathrm{kg}$.

C.4 Processed Data

An example of each required calculation is provided in this section using data point 1 from Table 5.

Eq. (23) needs the period T of one oscillation, not the time for ten. So, the period is calculated by dividing the time for ten oscillations by ten. However, while the timer used is accurate to the nearest hundredth of a second, I (who started and stopped the timer) am not nearly as fast. Therefore, the time for each oscillation could only be precise to the nearest tenth of a second.

$$10T = 26.9\,\mathrm{s} \pm 0.1\,\mathrm{s} \quad \Rightarrow \quad T = 2.69\,\mathrm{s} \pm 0.01\,\mathrm{s} \tag{24}$$

Next, an average period is calculated by averaging the individual periods. The uncertainty of this average is calculated by determining the range uncertainty of the data set.

$$\begin{aligned} \overline{T} &= \frac{2 \cdot 2.69\,\mathrm{s} + 3 \cdot 2.70\,\mathrm{s}}{5} \\ &= 2.70\,\mathrm{s} \end{aligned} \qquad \begin{aligned} U_{\overline{T}} &= \frac{T_{max} - T_{min}}{2} \\ &= \frac{2.70\,\mathrm{s} - 2.69\,\mathrm{s}}{2} = 0.01\,\mathrm{s} \end{aligned} \tag{25}$$

The average period $\overline{T}$ is $2.70\,\mathrm{s} \pm 0.01\,\mathrm{s}$.

Table 6: Period of Driven Gear in Bifilar Pendulum

Data Point #	**Period (s ± 0.01 s)**
1	2.69
2	2.69
3	2.70
4	2.70
5	2.70
Average	2.70

Using Eq. (23), the moment of inertia of the driven gear I_o could be calculated:

$$\begin{aligned} I_o &= \frac{m_o \vec{g} b^2 \overline{T}^2}{4\pi^2 L} \\ &= \frac{(0.55110\,\mathrm{kg})\,(9.81\,\mathrm{m\,s^{-2}})\,(0.0762\,\mathrm{m})^2\,(2.70\,\mathrm{s})^2}{4\pi^2\,(0.562\,\mathrm{m})} \\ &= 0.00257\,\mathrm{kg\,m^2} \end{aligned} \tag{26}$$

$$\begin{aligned} U_{rel\,I_o} &= U_{rel\,m_c} + 2 \cdot U_{rel\,b} + 2 \cdot U_{rel\,T} + U_{rel\,L} \\ &= \frac{0.00005\,\mathrm{kg}}{0.55110\,\mathrm{kg}} + 2 \cdot \frac{0.0005\,\mathrm{m}}{0.0762\,\mathrm{m}} + 2 \cdot \frac{0.01\,\mathrm{s}}{2.70\,\mathrm{s}} + \frac{0.001\,\mathrm{m}}{0.562\,\mathrm{m}} \\ &= 0.0224 \end{aligned} \tag{27}$$

$$U_{I_o} = I_o \cdot U_{rel\,I_o} = 0.00257\,\mathrm{kg\,m^2} \cdot 0.0224 = 0.00006\,\mathrm{kg\,m^2}$$

The moment of inertia of the driven gear I_o is determined to be $0.00257\,\mathrm{kg\,m^2} \pm 0.00006\,\mathrm{kg\,m^2}$.

D Data Processing Program

This program processses the raw data for use in the various other programs that follow. The raw dataset is embedded in this program so all the programs can be run on any computer.

```
from math import pi, sqrt, atan, floor, cos
import numpy as np
from scipy.interpolate import UnivariateSpline # v0.19.1 must be used


#
# PROCESS RAW DATA
#

length_of_arm = (0.824, 0.001/0.824)
width_of_arm = (0.0552, 0.0005/0.0552)
diagonal_of_half_arm = (sqrt( (length_of_arm[0]/2)**2 + (width_of_arm[0]/2)**2 ), 0.5 * (2 * length_of_arm[1]
     + 2 * width_of_arm[1]) )
phi = (atan( (width_of_arm[0]/2) / (length_of_arm[0]/2) ), 0.5 * width_of_arm[1] + 0.5 * length_of_arm[1])

n = 100000
period = []

#
# TRIAL 1
#

trial_1_data = np.array([[0.0, 0.0], [0.25, 0.11], [0.25, 0.41], [0.25, 0.88], [0.5, 1.5], [0.5, 2.18],
     [0.75, 2.87], [0.75, 3.64], [1.0, 4.8], [1.0, 5.4], [1.5, 6.97], [1.5, 7.86], [1.75, 8.58], [1.75,
     9.38], [2.0, 10.47], [2.0, 11.07], [2.25, 12.01], [2.5, 12.83], [2.5, 13.78], [2.75, 14.41], [2.75,
     15.43], [3.0, 15.74], [3.0, 16.51], [3.25, 16.97], [3.25, 17.78], [3.25, 17.92], [3.75, 20.37], [4.25,
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     39.81], [8.25, 41.42], [8.75, 43.34], [9.25, 44.86], [9.75, 46.15], [10.0, 47.71], [10.5, 48.88],
     [10.75, 49.62], [11.25, 50.35], [11.5, 51.33], [12.25, 52.53], [13.0, 53.77], [14.0, 55.11], [14.5,
     55.91], [14.75, 56.12], [15.0, 56.09], [15.25, 56.13], [15.25, 56.03], [15.5, 56.24], [15.75, 56.3],
     [16.0, 56.58], [16.25, 56.8], [16.75, 57.07], [17.25, 57.34], [17.75, 57.45], [18.0, 57.68], [18.5,
     57.88], [19.25, 57.83], [20.0, 58.16], [20.5, 57.93], [21.0, 58.13], [21.25, 58.05], [21.75, 58.05],
     [22.0, 57.9], [22.25, 57.81], [22.75, 57.84], [23.0, 57.81], [23.25, 57.78], [23.75, 57.76], [24.25,
     57.71], [24.5, 57.58], [24.75, 57.48], [25.25, 57.3], [25.5, 57.19], [25.75, 56.98], [26.0, 56.74],
     [26.25, 56.43], [26.5, 55.94], [26.75, 55.77], [27.0, 55.78], [27.25, 55.55], [27.5, 55.5], [28.25,
     55.38], [29.25, 54.0], [29.5, 52.36], [30.5, 52.13], [30.75, 50.52], [31.5, 49.5], [32.25, 47.31],
     [32.75, 45.68], [33.25, 43.57], [33.75, 41.0], [34.25, 38.12], [35.5, 34.63], [35.75, 32.84], [36.25,
     30.84], [37.0, 28.35], [37.25, 26.26], [37.75, 24.3], [38.25, 22.65], [38.5, 21.2], [39.0, 19.47],
     [39.25, 17.36], [39.75, 15.27], [40.25, 12.64], [40.5, 11.11], [41.0, 8.6], [41.5, 5.9], [42.25, 2.32],
     [42.25, 1.63], [42.75, -0.85], [42.75, -1.91], [43.25, -3.52], [43.5, -5.34], [44.0, -7.92], [44.25,
     -9.03], [44.5, -10.13], [44.5, -11.25], [45.0, -12.82], [45.25, -13.76], [46.0, -17.43], [46.0, -18.42],
     [46.5, -21.14], [47.25, -23.91], [47.75, -26.51], [48.5, -29.7], [49.0, -31.57], [49.5, -33.66], [50.25,
     -36.76], [50.75, -38.15], [51.25, -38.77], [52.0, -40.93], [52.75, -43.15], [53.5, -44.83], [54.25,
     -46.91], [54.75, -48.08], [55.5, -49.33], [56.0, -50.54], [56.5, -51.31], [56.5, -52.11], [56.75,
     -52.21], [57.5, -52.99], [58.25, -53.91], [59.25, -54.5], [60.5, -54.93], [61.0, -54.89], [61.75,
     -55.11], [62.25, -55.19], [63.0, -55.05], [63.25, -55.34], [63.75, -55.14], [65.5, -55.19], [67.0,
     -54.87], [68.5, -54.39], [69.0, -54.06], [70.25, -53.56], [71.0, -52.96], [72.0, -52.24], [72.5,
     -51.67], [73.75, -50.62], [74.0, -50.18], [76.5, -46.2], [77.25, -44.59], [77.75, -43.24], [78.25,
     -41.73], [78.75, -40.36], [79.25, -38.44], [79.75, -36.99], [80.5, -33.42], [81.25, -31.15], [82.0,
     -26.99], [82.5, -24.75], [83.0, -22.82]])
x1 = trial_1_data[:,0] # Difference in angular displacements (degrees)
y1 = trial_1_data[:,1] # Scale reading (grams)

x1 *= 0.0174533 # Convert degrees to radians
y1 = diagonal_of_half_arm[0] * (y1 * 0.00981) / cos(phi[0]) # Convert scale reading to torque
w1 = np.isnan(y1)
y1[w1] = 0

```

```
spl1 = UnivariateSpline(x1, y1, s=0.0004, w=~w1)
xs1 = np.linspace(0, np.pi/2, n)

# Truncate interpolation
cutoff_index_1 = 0
for i in range(floor(3*n/4), n):
    if spl1(xs1[i]) >= 0:
        cutoff_index_1 += 1
cutoff_index_1 -= 1
period.append(xs1[:-cutoff_index_1][-1])

# Trial 2

trial_2_data = np.array([[0.0, 0.0], [0.25, 0.14], [0.25, 0.51], [0.5, 0.98], [0.75, 1.55], [0.75, 2.34],
     [1.0, 3.05], [1.0, 3.67], [1.0, 4.86], [1.25, 5.41], [1.5, 7.09], [1.5, 7.95], [1.75, 8.67], [1.75,
     9.41], [2.0, 10.58], [2.0, 11.16], [2.25, 12.15], [2.75, 12.93], [2.75, 13.86], [2.75, 14.45], [2.75,
     15.56], [3.0, 15.85], [3.0, 16.62], [3.25, 17.15], [3.5, 17.97], [3.5, 18.08], [4.0, 20.44], [4.25,
     23.71], [4.75, 25.82], [5.75, 30.29], [6.25, 32.65], [6.5, 34.75], [7.0, 36.01], [7.75, 38.46], [7.75,
     39.89], [8.25, 41.49], [9.0, 43.46], [9.25, 44.93], [9.75, 46.35], [10.0, 47.89], [10.5, 48.93], [10.75,
     49.8], [11.25, 50.42], [11.5, 51.36], [12.25, 52.71], [13.0, 53.81], [14.0, 55.15], [14.5, 56.11],
     [15.0, 56.18], [15.0, 56.22], [15.25, 56.27], [15.25, 56.06], [15.75, 56.43], [15.75, 56.38], [16.0,
     56.59], [16.25, 56.85], [16.75, 57.09], [17.5, 57.44], [17.75, 57.58], [18.25, 57.81], [18.5, 58.04],
     [19.5, 57.95], [20.0, 58.21], [20.5, 58.04], [21.0, 58.31], [21.25, 58.12], [21.75, 58.23], [22.0,
     58.06], [22.25, 57.86], [22.75, 57.95], [23.0, 57.96], [23.25, 57.92], [24.0, 57.88], [24.25, 57.89],
     [24.5, 57.73], [25.0, 57.54], [25.25, 57.38], [25.5, 57.2], [25.75, 57.12], [26.25, 56.8], [26.25,
     56.48], [26.5, 56.02], [27.0, 55.84], [27.0, 55.86], [27.25, 55.63], [27.75, 55.55], [28.25, 55.52],
     [29.25, 54.18], [29.75, 52.5], [30.5, 52.2], [30.75, 50.56], [31.75, 49.62], [32.25, 47.35], [32.75,
     45.73], [33.25, 43.69], [33.75, 41.18], [34.25, 38.19], [35.5, 34.69], [35.75, 32.92], [36.25, 30.91],
     [37.0, 28.53], [37.25, 26.31], [38.0, 24.38], [38.25, 22.78], [38.5, 21.3], [39.0, 19.54], [39.25,
     17.41], [39.75, 15.42], [40.25, 12.71], [40.75, 11.29], [41.25, 8.69], [41.5, 6.0], [42.25, 2.45],
     [42.25, 1.77], [42.75, 0.71], [43.0, 0.23], [43.25, -0.87], [43.5, -1.93], [44.25, -3.53], [44.25,
     -5.37], [44.5, -8.02], [44.5, -9.21], [45.0, -10.14], [45.25, -11.38], [46.0, -12.97], [46.25, -13.9],
     [46.75, -17.56], [47.25, -18.43], [47.75, -21.29], [48.75, -24.07], [49.0, -26.65], [49.5, -29.77],
     [50.25, -31.59], [50.75, -33.69], [51.25, -36.91], [52.0, -38.24], [53.0, -38.88], [53.75, -41.08],
     [54.25, -43.21], [54.75, -44.94], [55.5, -46.96], [56.25, -48.14], [56.5, -49.33], [56.75, -50.56],
     [56.75, -51.33], [57.5, -52.16], [58.25, -52.24], [59.5, -53.01], [60.75, -53.99], [61.0, -54.7],
     [61.75, -55.03], [62.25, -54.99], [63.0, -55.26], [63.25, -55.36], [63.75, -55.25], [65.5, -55.41],
     [67.25, -55.2], [68.75, -55.31], [69.0, -55.0], [70.25, -54.5], [71.25, -54.22], [72.0, -53.64], [72.5,
     -53.05], [73.75, -52.32], [74.25, -51.8], [76.5, -50.66], [77.25, -50.27], [78.0, -46.35], [78.25,
     -44.64], [78.75, -43.42], [79.25, -41.85], [79.75, -40.37], [80.5, -38.51], [81.0, -37.09], [81.25,
     -33.57], [81.5, -31.21]])
x2 = trial_2_data[:,0] # Difference in angular displacements (degrees)
y2 = trial_2_data[:,1] # Scale reading (grams)

x2 *= 0.0174533 # Convert degrees to radians
y2 = diagonal_of_half_arm[0] * (y2 * 0.00981) / cos(phi[0]) # Convert scale reading to torque
w2 = np.isnan(y2)
y2[w2] = 0

spl2 = UnivariateSpline(x2, y2, s=0.002, w=~w2)
xs2 = np.linspace(0, np.pi/2, n)

# Truncate interpolation
cutoff_index_2 = 0
for i in range(floor(3*n/4), n):
    if spl2(xs2[i]) > 0:
        cutoff_index_2 += 1
cutoff_index_2 -= 1
period.append(xs2[:-cutoff_index_2][-1])

# Trial 3

trial_3_data = np.array([[0.0, 0.0], [0.25, 0.17], [0.25, 0.49], [0.5, 0.88], [0.75, 1.51], [0.75, 2.26],
     [1.0, 3.16], [1.0, 3.7], [1.0, 4.98], [1.25, 5.59], [1.5, 7.2], [1.75, 8.02], [2.0, 8.82], [2.0, 9.56],
     [2.0, 10.73], [2.25, 11.21], [2.5, 12.18], [2.75, 13.04], [2.75, 13.98], [3.0, 14.54], [3.0, 15.68],
     [3.25, 16.02], [3.25, 16.8], [3.5, 17.19], [3.5, 18.16], [3.5, 18.15], [4.0, 20.53], [4.25, 23.82],
     [5.0, 25.98], [6.0, 30.47], [6.25, 32.8], [6.5, 34.77], [7.0, 36.01], [7.75, 38.64], [8.0, 39.98], [8.5,
     41.52], [9.0, 43.5], [9.5, 45.1], [10.0, 46.42], [10.0, 47.92], [10.5, 49.05], [11.0, 49.9], [11.5,
```

```
    50.56], [11.5, 51.5], [12.5, 52.83], [13.25, 53.86], [14.0, 55.16], [14.75, 56.16], [15.0, 56.3], [15.0,
    56.32], [15.5, 56.36], [15.5, 56.19], [16.0, 56.52], [15.75, 56.52], [16.0, 56.68], [16.5, 56.87],
    [17.0, 57.28], [17.5, 57.63], [17.75, 57.73], [18.5, 58.0], [18.5, 58.05], [19.5, 58.1], [20.25, 58.21],
    [20.5, 58.06], [21.0, 58.35], [21.25, 58.29], [22.0, 58.24], [22.25, 58.13], [22.5, 58.04], [22.75,
    58.0], [23.0, 57.98], [23.25, 57.95], [24.0, 58.06], [24.5, 58.09], [24.5, 57.86], [25.0, 57.59],
    [25.25, 57.41], [25.5, 57.2], [25.75, 57.25], [26.25, 56.8], [26.5, 56.62], [26.5, 56.18], [27.0,
    55.98], [27.0, 55.94], [27.5, 55.76], [27.75, 55.57], [28.5, 55.58], [29.5, 54.36], [29.75, 52.69],
    [30.5, 52.3], [31.0, 50.64], [32.0, 49.73], [32.25, 47.44], [33.0, 45.8], [33.25, 43.83], [33.75,
    41.26], [34.25, 38.25], [35.75, 34.72], [35.75, 33.08], [36.5, 31.09], [37.0, 28.62], [37.5, 26.39],
    [38.0, 24.48], [38.25, 22.82], [38.5, 21.43], [39.0, 19.63], [39.5, 17.54], [40.0, 15.52], [40.5,
    12.74], [41.0, 11.32], [41.25, 8.87], [41.5, 6.17], [42.5, 2.62], [42.5, 1.79], [43.0, 0.83], [43.25,
    -0.94], [43.25, -2.05], [43.5, -3.54], [44.5, -5.48], [44.25, -7.99], [44.5, -9.05], [44.75, -10.24],
    [45.0, -11.44], [45.25, -12.92], [46.0, -13.8], [46.25, -17.5], [47.0, -18.45], [47.5, -21.16], [47.75,
    -24.05], [48.75, -26.55], [49.25, -29.79], [49.5, -31.7], [50.5, -33.73], [51.0, -36.92], [51.25,
    -38.26], [52.0, -38.88], [53.25, -41.05], [54.0, -43.2], [54.5, -44.94], [55.0, -46.93], [55.5, -48.17],
    [56.25, -49.52], [56.5, -50.66], [56.75, -51.51], [56.75, -52.27], [57.5, -52.37], [58.25, -53.05],
    [59.5, -54.1], [60.75, -54.6], [61.0, -54.96], [61.75, -55.09], [62.5, -55.28], [63.0, -55.31], [63.5,
    -55.22], [63.75, -55.43], [65.75, -55.14], [67.25, -55.36], [68.75, -55.02], [69.0, -54.46], [70.25,
    -54.17], [71.25, -53.73], [72.25, -53.12], [72.5, -52.36], [73.75, -51.78], [74.25, -50.66], [76.5,
    -50.21], [77.5, -46.23], [78.25, -44.71], [78.5, -43.36], [78.75, -41.79], [79.5, -40.4], [79.75,
    -38.51], [80.5, -37.15], [81.5, -33.42], [82.0, -31.32], [82.5, -27.01]])
x3 = trial_3_data[:,0] # Difference in angular displacements (degrees)
y3 = trial_3_data[:,1] # Scale reading (grams)

x3 *= 0.0174533 # Convert degrees to radians
y3 = diagonal_of_half_arm[0] * (y3 * 0.00981) / cos(phi[0]) # Convert scale reading to torque
w3 = np.isnan(y3)
y3[w3] = 0

spl3 = UnivariateSpline(x3, y3, s=0.0015, w=~w3)
xs3 = np.linspace(0, np.pi/2, n)

# Truncate interpolation
cutoff_index_3 = 0
for i in range(floor(3*n/4), n):
    if spl3(xs3[i]) > 0:
        cutoff_index_3 += 1
cutoff_index_3 -= 1
period.append(xs3[:-cutoff_index_3][-1])


average_period = sum(period)/float(len(period))
period_unc = round((max(period) - min(period)) / 2, 2)
```

E Sine Wave-Fitting Program

This program attempts to fit a sine wave to the processed data from Trial 1 of the torque curve generation and outputs the result on a plot. It is based on the algorithm suggested by StackOverflow user Dhara (2013).

```
import numpy as np
from scipy.optimize import leastsq
import matplotlib
matplotlib.use('TkAgg')
import pylab as plt
from math import sqrt, atan, cos
from process_data import *

guess_mean = np.mean(y1)/2
guess_std = 3*np.std(y1)/(2**0.5)
guess_phase = 0
guess_stretch = 0.3


data_first_guess = guess_std*np.sin(np.sin(guess_stretch**-1 * (x1))) + guess_mean

optimize_func = lambda x: x[0]*np.sin(np.sin(x[1]**-1 *(x1))) - y1
est_std, est_stretch, est_mean = leastsq(optimize_func, [guess_std, guess_stretch, guess_mean])[0]

fig = plt.figure(1, figsize=(9, 5), dpi=150)
fig.suptitle('\\textbf{Torque Felt by Driven Gear vs. Difference in Displacements}', fontweight='bold')
fig.subplots_adjust(left=0.11, top=0.9, right=0.98, bottom=0.1)

plt.plot(x1, y1, '.', label='Processed Data Points', c='black')
plt.plot(x1, est_std*np.sin(est_stretch**-1 *(x1)+est_mean), '--', label='Fitted Sine Wave', c='black')

plt.ylabel('\\textbf{Torque Felt by\\\\Driven Gear (Nm)}')
plt.xlabel('\\textbf{Difference in Displacements (rad)}')
plt.xlim(0, np.pi/2)

plt.legend(numpoints=1)
plt.show()
```

F Spline Interpolation Program

This is the program fits spline interpolations to each trial of the torque curve generation and outputs the result on a plot.

```
from scipy.interpolate import UnivariateSpline # v0.19.1 must be used
import numpy as np
from math import cos, sqrt, atan, floor
import matplotlib
matplotlib.use('TkAgg')
import pylab as plt
import sys
from process_data import *

print('The periods of each spline interpolation are {}, respectively.'.format(period))

fig = plt.figure(1, figsize=(9, 7), dpi=150)
fig.subplots_adjust(hspace=0.76,left=0.11, top=0.94, right=0.74, bottom=0.1)
anchor = (1.4, 1.02)

ax1 = plt.subplot(311)
ax1.plot(x1, y1, '.', c='black', label='Processed\nData Points')
ax1.plot(xs1[:-cutoff_index_1], spl1(xs1[:-cutoff_index_1]), '--', c='black', lw=1,
      label='Spline\nInterpolation')
plt.ylabel('\\textbf{Torque Felt by\\\\Driven Gear (Nm)}')
plt.xlabel('\\textbf{Difference in Displacements (rad)}')
plt.legend(numpoints=1,bbox_to_anchor=anchor)
ax1.set_title('\\textbf{Spline Interpolation for Trial 1}')

ax2 = plt.subplot(312)
ax2.plot(x2, y2, '*', c='black', label='Processed\nData Points')
ax2.plot(xs2[:-cutoff_index_2], spl2(xs2[:-cutoff_index_2]), '--', c='black', lw=1,
      label='Spline\nInterpolation')
plt.ylabel('\\textbf{Torque Felt by\\\\Driven Gear (Nm)}')
plt.xlabel('\\textbf{Difference in Displacements (rad)}')
plt.legend(numpoints=1,bbox_to_anchor=anchor)
ax2.set_title('\\textbf{Spline Interpolation for Trial 2}')

ax3 = plt.subplot(313)
plt.plot(x3, y3, 'x', c='black', label='Processed\nData Points')
plt.plot(xs3[:-cutoff_index_3], spl3(xs3[:-cutoff_index_3]), '--', c='black', lw=1,
      label='Spline\nInterpolation')
plt.ylabel('\\textbf{Torque Felt by\\\\Driven Gear (Nm)}')
plt.xlabel('\\textbf{Difference in Displacements (rad)}')
plt.legend(numpoints=1,bbox_to_anchor=anchor)
ax3.set_title('\\textbf{Spline Interpolation for Trial 3}')

plt.savefig('processed_w_fit.png')
plt.show()
```

G Runge-Kutta Program

This is the main Runge-Kutta program that predicts the behaviour of the driven gear as outlined in Section 3.

```
# Runge-Kutta Method
# Script written by Adam Vandenbussche on October 8, 2017
# Based on the method outlined in Elementary Differential Equations with Applications 2nd ed. (Derrick &
    Grossman, 1981)
#
# Run like this:
# python runge-kutta.py a+-a_unc t h I_0+-I_0_unc I_1+-I_1_unc I_2+-I_2_unc --ignore-tolerance
# where: a+-a_unc -> Driver acceleration and absolute uncertainty
#        t -> Total time interval to compute
#        h -> Runge-Kutta increment
#        I_0+-I_0_unc -> Coefficient of velcoity^0 of driven gear (moment of inertia) and absolute uncertainty
#        I_1+-I_1_unc -> Coefficient of velcoity^1 of driven gear (moment of inertia) and absolute uncertainty
#        I_2+-I_2_unc -> Coefficient of velcoity^2 of driven gear (moment of inertia) and absolute uncertainty
#        --ignore-tolerance -> Ignore maximum displacement per interval warning
#


# Import dependencies
from math import pi, ceil, log
import numpy as np
import matplotlib
matplotlib.use('TkAgg')
import pylab as plt
import sys
from process_data import *

def main(args):

    #
    # PROCESS PROGRAM PARAMETERS
    #

    h = float(args[3]) # Step size
    total_time = float(args[2]) # Seconds
    current_time = 0

    # Driver acceleration
    driver_a, driver_a_unc = args[1].split('+-')
    driver_a = float(driver_a)
    driver_a_unc = float(driver_a_unc)

    total_provided_unc = driver_a_unc / driver_a

    driver_v = 0

    # Moment of inertia quadratic coefficients
    moi_0, moi_0_unc = args[4].split('+-')
    moi_0 = float(moi_0)
    moi_0_unc = float(moi_0_unc)

    if moi_0 != 0:
        total_provided_unc += moi_0_unc / moi_0

    moi_1, moi_1_unc = args[5].split('+-')
    moi_1 = float(moi_1)
    moi_1_unc = float(moi_1_unc)

    if moi_1 != 0:
```

```
        total_provided_unc += moi_1_unc / moi_1

    moi_2, moi_2_unc = args[6].split('+-')
    moi_2 = float(moi_2)
    moi_2_unc = float(moi_2_unc)

    if moi_2 != 0:
        total_provided_unc += moi_2_unc / moi_2


    #
    # MAIN RUNGE-KUTTA ALGORITHM
    #

    iterations = ceil(total_time/h)
    successful_iterations = 0
    already_alerted = False

    time_xs = np.empty([1, iterations])
    displacement_ys = np.empty([1, iterations])
    velocity_ys = np.empty([1, iterations]) ######
    acceleration_ys = np.empty([1, iterations])
    acceleration_ys_unc = np.empty([1, iterations])
    velocity_ys_unc = np.empty([1, iterations])
    displacement_ys_unc = np.empty([1, iterations])

    def driven_acceleration(time, driven_displacement, driven_velocity):
        delta_theta = 0.5 * driver_a * time ** 2 - driven_displacement
        values = [ spl1( delta_theta % period[0]), spl2( delta_theta % period[1]), spl3( delta_theta % period[2]) ]

        average = sum(values) / 3
        range_unc = (max(values) - min(values)) / 2

        moi = moi_0 + moi_1 * driven_velocity + moi_2 * driven_velocity ** 2

        return (average / moi, range_unc / moi, [values[0]/moi, values[1]/moi, values[2]/moi])

    # Initial conditions
    d_current = 0
    v_current = 0

    for i in range(iterations):
        #print('Performing calculation {0:,d} of {1:,d}!'.format(i, iterations))
        driver_v += driver_a * h
        driver_d = 0.5 * driver_a * h ** 2
        if (driver_v * h + driver_d > pi/24):
            if not already_alerted:
                print('Surpassed maximum displacement per iteration threshold after {0} seconds! Stopping simulation.'.format(current_time))
                already_alerted = True
            if '--ignore-tolerance' not in args:
                break

        time_xs[0][i] = current_time
        displacement_ys[0][i] = d_current
        velocity_ys[0][i] = v_current

        dd1 = h * v_current
        dv1, dv1_a_unc, dv1_a_individual = driven_acceleration(current_time, d_current, v_current)
        dv1 *= h

        dd2 = h * (v_current + dv1 / 2)
        dv2, dv2_a_unc, dv2_a_individual = driven_acceleration(current_time + h / 2, d_current + dd1 / 2, v_current + dv1 / 2)
        dv2 *= h

        dd3 = h * (v_current + dv2 / 2)
```

```
        dv3, dv3_a_unc, dv3_a_individual = driven_acceleration(current_time + h / 2, d_current + dd2 / 2, v_current + dv2 / 2)
        dv3 *= h

        dd4 = h * (v_current + dv3)
        dv4, dv4_a_unc, dv4_a_individual = driven_acceleration(current_time + h, d_current + dd3, v_current + dv3)
        dv4 *= h

        dd = (dd1 + 2 * dd2 + 2 * dd3 + dd4) / 6
        dv = (dv1 + 2 * dv2 + 2 * dv3 + dv4) / 6
        d_current += dd
        v_current += dv

        acceleration_ys[0][i], acceleration_ys_unc[0][i] = driven_acceleration(current_time + h, d_current, v_current)[:2]

        velocity_ys_unc[0][i] = h * acceleration_ys_unc[0][i]
        displacement_ys_unc[0][i] = h * velocity_ys_unc[0][i] + 0.5 * acceleration_ys_unc[0][i] * (h ** 2)

        current_time += h
        successful_iterations += 1


    difference_ys = np.empty([1, successful_iterations])
    difference_ys_unc = np.empty([1, successful_iterations])

    # Analyze slipping
    done_analyzing_slipping = False
    slipping_time = 0
    earliest_possible_slipping_time = 0
    latest_possible_slipping_time = 0
    for i in range(successful_iterations):
        difference_ys[0][i] = 0.5 * driver_a * (h * i) ** 2 - displacement_ys[0][i]
        difference_ys_unc[0][i] = driver_a_unc / driver_a + displacement_ys_unc[0][i]
        if not done_analyzing_slipping and earliest_possible_slipping_time == 0 and difference_ys[0][i] * (1 + total_provided_unc) + difference_ys_unc[0][i] > average_period - period_unc:
            earliest_possible_slipping_time = h * i
        if not done_analyzing_slipping and slipping_time == 0 and difference_ys[0][i] > average_period:
            slipping_time = h * i
        if not done_analyzing_slipping and difference_ys[0][i] * (1 - total_provided_unc) - difference_ys_unc[0][i] > average_period + period_unc:
            done_analyzing_slipping = True
            latest_possible_slipping_time = h * i
    if done_analyzing_slipping:
        print('Slipping occurs after {0} s, although could occur between {1} and {2} s ({0}+{3}-{4}).'.format(slipping_time, earliest_possible_slipping_time, latest_possible_slipping_time, slipping_time-earliest_possible_slipping_time, latest_possible_slipping_time-slipping_time))
    else:
        print('No slipping occurs!')

    # Determine peak driven gear acceleration
    max_driven_a = np.amax(acceleration_ys)
    max_driven_a_unc = max_driven_a * total_provided_unc + acceleration_ys_unc[0][np.argmax(acceleration_ys)]
    print('Peak Driven Acceleration: {} Âś {}'.format( max_driven_a, max_driven_a_unc))



    #
    # GENERATE PLOTS
    #

    # Set up window
    fig = plt.figure(1, figsize=(9, 6), dpi=150)
    fig.subplots_adjust(hspace=0.48, wspace=0.24, left=0.1, top=0.94, right=0.98, bottom=0.18)
    size = 0.2 ** 2 # Size of points in pt^2
    x_time = np.linspace(0, h * successful_iterations, successful_iterations)
```

```
    opacity = 0.2

    # Acceleration-Time
    ax1 = plt.subplot(221)
    # Driver uncertainty
    unc = ax1.fill_between(time_xs[0][:successful_iterations - 2], np.linspace(driver_a, driver_a, successful_iterations - 2) + np.linspace(driver_a_unc, driver_a_unc, successful_iterations - 2), np.linspace(driver_a, driver_a, successful_iterations - 2) - np.linspace(driver_a_unc, driver_a_unc, successful_iterations - 2), facecolor='black', edgecolor='none', alpha=opacity, interpolate=True)
    # Driven uncertainty
    ax1.fill_between(time_xs[0][:successful_iterations - 2], acceleration_ys[0][:successful_iterations - 2] * (1 + total_provided_unc) + acceleration_ys_unc[0][:successful_iterations - 2], acceleration_ys[0][:successful_iterations - 2] * (1 - total_provided_unc) - acceleration_ys_unc[0][:successful_iterations - 2], facecolor='black', edgecolor='none', alpha=opacity, interpolate=True)
    # Driver
    ax1.plot(x_time, np.linspace(driver_a, driver_a, successful_iterations), '--', c='black', label='Driver Acceleration')
    # Driven
    ax1.scatter(time_xs[:,:successful_iterations - 2], acceleration_ys[:,:successful_iterations - 2], s=0.01, marker='.', color='black', label='Driven Acceleration', alpha=0.5+log(h, 10)*0.04)

    plt.ylabel('\\textbf{Acceleration (rad s\\textsuperscript{-2})}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, (successful_iterations - 2) * h)
    ax1.set_title('\\textbf{Acceleration vs. Time}')


    # Velocity-Time
    ax2 = plt.subplot(222)
    # Driver uncertainty
    ax2.fill_between(x_time, (np.linspace(driver_a, driver_a, successful_iterations) * x_time) * (1 + np.linspace(driver_a_unc/driver_a, driver_a_unc/driver_a, successful_iterations)), (np.linspace(driver_a, driver_a, successful_iterations) * x_time) * (1 - np.linspace(driver_a_unc/driver_a, driver_a_unc/driver_a, successful_iterations)), facecolor='black', edgecolor='none', alpha=opacity, interpolate=True)
    # Driven uncertainty
    ax2.fill_between(time_xs[0][:successful_iterations], velocity_ys[0][:successful_iterations] * (1 + total_provided_unc) + velocity_ys_unc[0][:successful_iterations], velocity_ys[0][:successful_iterations] * (1 - total_provided_unc) - velocity_ys_unc[0][:successful_iterations], facecolor='black', edgecolor='none', alpha=opacity, interpolate=True)

    driver = ax2.plot(x_time, driver_a*x_time, '--', c='black', label='Driver Velocity')
    driven = ax2.scatter(time_xs[:,:successful_iterations], velocity_ys[:,:successful_iterations], s=0.005, marker='.', color='black', label='Driven Velocity')

    plt.ylabel('\\textbf{Velocity (rad s\\textsuperscript{-1})}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, successful_iterations * h)
    ax2.set_title('\\textbf{Velocity vs. Time}')


    # Difference in Displacements-Time
    ax3 = plt.subplot(223)

    ax3.fill_between(time_xs[0][:successful_iterations], difference_ys[0][:successful_iterations] * (1 + total_provided_unc) + difference_ys_unc[0][:successful_iterations], difference_ys[0][:successful_iterations] * (1 - total_provided_unc) - difference_ys_unc[0][:successful_iterations], facecolor='black', edgecolor='none', alpha=opacity, interpolate=True)

    ax3.scatter(time_xs[:,:successful_iterations], difference_ys[0], s=0.01, marker='.', color='black', label='Difference in Displacements')
    plt.ylabel('\\textbf{Difference in \\\\ Displacements (rad)}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, successful_iterations * h)
    ax3.set_title('\\textbf{Difference in Displacements vs. Time}')
```

```python


    # Displacement-Time
    ax4 = plt.subplot(224)
    # Driver uncertainty
    ax4.fill_between(x_time, (0.5 * np.linspace(driver_a, driver_a, successful_iterations) * x_time ** 2) *
        (1 + np.linspace(2*driver_a_unc/driver_a, 2*driver_a_unc/driver_a, successful_iterations)), (0.5 *
        np.linspace(driver_a, driver_a, successful_iterations) * x_time ** 2) * (1 -
        np.linspace(2*driver_a_unc/driver_a, 2*driver_a_unc/driver_a, successful_iterations)),
        facecolor='black', edgecolor='none', alpha=opacity, interpolate=True)
    # Driven uncertainty
    ax4.fill_between(time_xs[0][:successful_iterations], displacement_ys[0][:successful_iterations] * (1 +
        total_provided_unc) + displacement_ys_unc[0][:successful_iterations],
        displacement_ys[0][:successful_iterations] * (1 - total_provided_unc) -
        displacement_ys_unc[0][:successful_iterations], facecolor='black', edgecolor='none', alpha=opacity,
        interpolate=True)

    ax4.plot(x_time, 0.5*driver_a*x_time**2, '--', c='black', label='Driver Displacement')
    ax4.scatter(time_xs[:,:successful_iterations], displacement_ys[:,:successful_iterations], s=size,
        marker='.', c='black', label='Driven Displacement')
    plt.ylabel('\\textbf{Displacement (rad)}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, successful_iterations * h)
    ax4.set_title('\\textbf{Displacement vs. Time}')


    # Custom legend
    labels = ['Driver Gear', 'Driven Gear']
    markers = ['_', '.']
    colors = ['black', 'black']
    patches = [ plt.plot([],[], marker=markers[i], ms=10, ls='', mec=None, color=colors[i],
        label='{:s}'.format(labels[i]) )[0] for i in range(len(labels)) ]
    labels.append('Uncertainty')
    patches.append(unc)

    fig.legend(patches, labels, numpoints=1, loc='lower center', ncol=3)

    # Generate separate velocity-time and displacemen-time graphs for driver and driven to emphasize that
        they are the same

    # Velocity-Time
    fig2 = plt.figure(2, figsize=(9, 2.5), dpi=150)
    fig2.subplots_adjust(hspace=0.48, wspace=0.24, left=0.1, top=0.9, right=0.98, bottom=0.17)
    axA = plt.subplot(121)
    # Driver uncertainty
    axA.fill_between(x_time, (np.linspace(driver_a, driver_a, successful_iterations) * x_time) * (1 +
        np.linspace(driver_a_unc/driver_a, driver_a_unc/driver_a, successful_iterations)),
        (np.linspace(driver_a, driver_a, successful_iterations) * x_time) * (1 -
        np.linspace(driver_a_unc/driver_a, driver_a_unc/driver_a, successful_iterations)), facecolor='black',
        edgecolor='none', alpha=opacity, interpolate=True)

    driver = axA.plot(x_time, driver_a*x_time, '--', c='black', label='Driver Velocity')

    plt.ylabel('\\textbf{Velocity (rad s\\textsuperscript{-1})}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, successful_iterations * h)
    axA.set_title('\\textbf{Driver Gear Velocity vs. Time}')

    axB = plt.subplot(122)

    # Driven uncertainty
    axB.fill_between(time_xs[0][:successful_iterations], velocity_ys[0][:successful_iterations] * (1 +
        total_provided_unc) + velocity_ys_unc[0][:successful_iterations],
        velocity_ys[0][:successful_iterations] * (1 - total_provided_unc) -
        velocity_ys_unc[0][:successful_iterations], facecolor='black', edgecolor='none', alpha=opacity,
        interpolate=True)

    driven = axB.scatter(time_xs[:,:successful_iterations], velocity_ys[:,:successful_iterations], s=0.005,
```

```
        marker='.', color='black', label='Driven Velocity')

    plt.ylabel('\\textbf{Velocity (rad s\\textsuperscript{-1})}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, successful_iterations * h)
    axB.set_title('\\textbf{Driven Gear Velocity vs. Time}')




    fig3 = plt.figure(3, figsize=(9, 2.5), dpi=150)
    fig3.subplots_adjust(hspace=0.48, wspace=0.24, left=0.1, top=0.9, right=0.98, bottom=0.17)

    # Displacement-Time
    axC = plt.subplot(121)
    # Driver uncertainty
    axC.fill_between(x_time, (0.5 * np.linspace(driver_a, driver_a, successful_iterations) * x_time ** 2) *
        (1 + np.linspace(2*driver_a_unc/driver_a, 2*driver_a_unc/driver_a, successful_iterations)), (0.5 *
        np.linspace(driver_a, driver_a, successful_iterations) * x_time ** 2) * (1 -
        np.linspace(2*driver_a_unc/driver_a, 2*driver_a_unc/driver_a, successful_iterations)),
        facecolor='black', edgecolor='none', alpha=opacity, interpolate=True)

    axC.plot(x_time, 0.5*driver_a*x_time**2, '--', c='black', label='Driver Displacement')
    plt.ylabel('\\textbf{Displacement (rad)}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, successful_iterations * h)
    axC.set_title('\\textbf{Driver Gear Displacement vs. Time}')

    # Displacement-Time
    axD = plt.subplot(122)
    # Driven uncertainty
    axD.fill_between(time_xs[0][:successful_iterations], displacement_ys[0][:successful_iterations] * (1 +
        total_provided_unc) + displacement_ys_unc[0][:successful_iterations],
        displacement_ys[0][:successful_iterations] * (1 - total_provided_unc) -
        displacement_ys_unc[0][:successful_iterations], facecolor='black', edgecolor='none', alpha=opacity,
        interpolate=True)

    axD.scatter(time_xs[:,:successful_iterations], displacement_ys[:,:successful_iterations], s=size,
        marker='.', c='black', label='Driven Displacement')
    plt.ylabel('\\textbf{Displacement (rad)}')
    plt.xlabel('\\textbf{Time Elapsed (s)}')
    plt.xlim(0, successful_iterations * h)
    axD.set_title('\\textbf{Driven Gear Displacement vs. Time}')

    plt.show()

if __name__=='__main__':
    sys.exit(main(sys.argv))
```

G.1 Validation of Program

One way we can convince ourselves of the accuracy of this program is by looking at the peak driven acceleration suggested by the program: this value should not exceed the quotient of the maximum torque applied by the driver gear on the driven gear (the maximum of the $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function) and the moment of inertia.

The average maximum value for $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ across all trials (determined programmatically) and its uncertainty (calculated using a range uncertainty) is provided in Table 7 using the simulation from Figure 11 of Section 4.

Table 7: Peak Driven Accelerations

Trial #	$\mathbf{\Delta\vec{\Theta}}$ **(rad ± rad)**	**Peak** $\vec{\mathrm{T}}(\mathbf{\Delta\vec{\Theta}})$ **(N m ± N m)**
1	0.349 ± 0.008	0.235 ± 0.009
2	0.367 ± 0.008	0.236 ± 0.009
3	0.367 ± 0.008	0.236 ± 0.009
Average	0.361 ± 0.009	0.236 ± 0.001

The maximum possible driven gear acceleration is calculated using Newton's second law:

$$\vec{\alpha}_{o_{peak}} = \frac{\vec{\mathrm{T}}(\Delta\vec{\Theta})_{peak}}{I_o} \approx \frac{0.236\,\mathrm{N\,m}}{0.00257\,\mathrm{kg\,m^2}} \approx 91.8\,\mathrm{rad\,s^{-2}}$$

$$\begin{aligned} U_{\vec{\alpha}_{o_{peak}}} &= \vec{\alpha}_{o_{peak}} \left(\frac{U_{\vec{\mathrm{T}}(\Delta\vec{\Theta})_{peak}}}{\vec{\mathrm{T}}(\Delta\vec{\Theta})_{peak}} + \frac{U_{I_o}}{I_o} \right) \\ &= 91.8\,\mathrm{rad\,s^{-2}} \left(\frac{0.001\,\mathrm{N\,m}}{0.236\,\mathrm{N\,m}} + \frac{0.00006\,\mathrm{kg\,m^2}}{0.00257\,\mathrm{kg\,m^2}} \right) \\ &\approx 3\,\mathrm{rad\,s^{-2}} \end{aligned} \tag{28}$$

The simulated peak driven gear acceleration, rounded to one significant figure, was $92\,\mathrm{rad\,s^{-2}}$ $\pm\ 3\,\mathrm{rad\,s^{-2}}$ (determined by the Runge-Kutta program). This value matches the expected maximum value calculated in Eq. (28). The fact that these values match suggests the program is at least partly accurate.

H Other Types of Accelerating Systems

H.1 System with a Quadratic Moment of Inertia

When factors such as friction and air resistance cannot be ignored, the moment of inertia of an accelerating object is not constant[21]: the faster an object spins, the more air resistance is generated, the more force required to continue to accelerate the object, the "heavier" the object feels (recalling that moment of inertia can be thought of as "angular mass").

Fundamentally, the resistive torques that are generated when objects begin rotating, especially at higher speeds, can have one of three behaviours (MathWorks, "MATLAB"): they can be constant, or they can be proportional to either angular velocity or angular velocity squared. This being said, should one decide they want to take various resistive torques into account as the gear train accelerates, they can model these torques as moments of inertia that increase proportional to angular velocity, angular velocity squared, etc.

In other words, Eq. (5) can be rewritten as follows:

$$\Delta\ddot{\vec{\theta}}_o = \frac{\vec{\mathrm{T}}\left(\frac{\vec{\alpha}_i \Delta t^2}{2} - \Delta\vec{\theta}_o\right)}{I_o} = \frac{\vec{\mathrm{T}}\left(\frac{\vec{\alpha}_i \Delta t^2}{2} - \Delta\vec{\theta}_o\right)}{\iota_0 + \iota_1 \Delta\dot{\vec{\theta}}_o + \iota_2 \Delta\dot{\vec{\theta}}_o^2} \tag{29}$$

where ι_n is the coefficient for the term $\Delta\dot{\vec{\theta}}_o^n$[22].

Furthermore, the uncertainty associated with the driven acceleration as shown in Eq. (30) is now calculated as follows:

$$U_{\Delta\ddot{\vec{\theta}}_o} = U_{\vec{\mathrm{T}}(\Delta\vec{\Theta})} + \Delta\ddot{\vec{\theta}}_o\left(\frac{U_{\vec{\alpha}_i}}{\vec{\alpha}_i} + \frac{U_{I_o}}{I_o}\right) = U_{\vec{\mathrm{T}}(\Delta\vec{\Theta})} + \Delta\ddot{\vec{\theta}}_o\left(\frac{U_{\vec{\alpha}_i}}{\vec{\alpha}_i} + \frac{U_{\iota_0}}{\iota_0} + \frac{U_{\iota_1}}{\iota_1} + \frac{U_{\iota_2}}{\iota_2}\right) \tag{30}$$

The same approach that discussed in Section 4 can be used to determine when slipping occurs in a magnetic gear train in the real world, that is, where air friction and rolling resistance are significant. Consider a magnetic gear train with the $\vec{\mathrm{T}}(\Delta\vec{\Theta})$ function determined in Section 2.1 and a moment of inertia represented by the quadratic function

$$(2.57 \pm 0.06) \cdot 10^{-3} + (2.00 \pm 0.02) \cdot 10^{-5}\Delta\dot{\vec{\theta}}_o + (5.00 \pm 0.05) \cdot 10^{-5}\Delta\dot{\vec{\theta}}_o^2$$

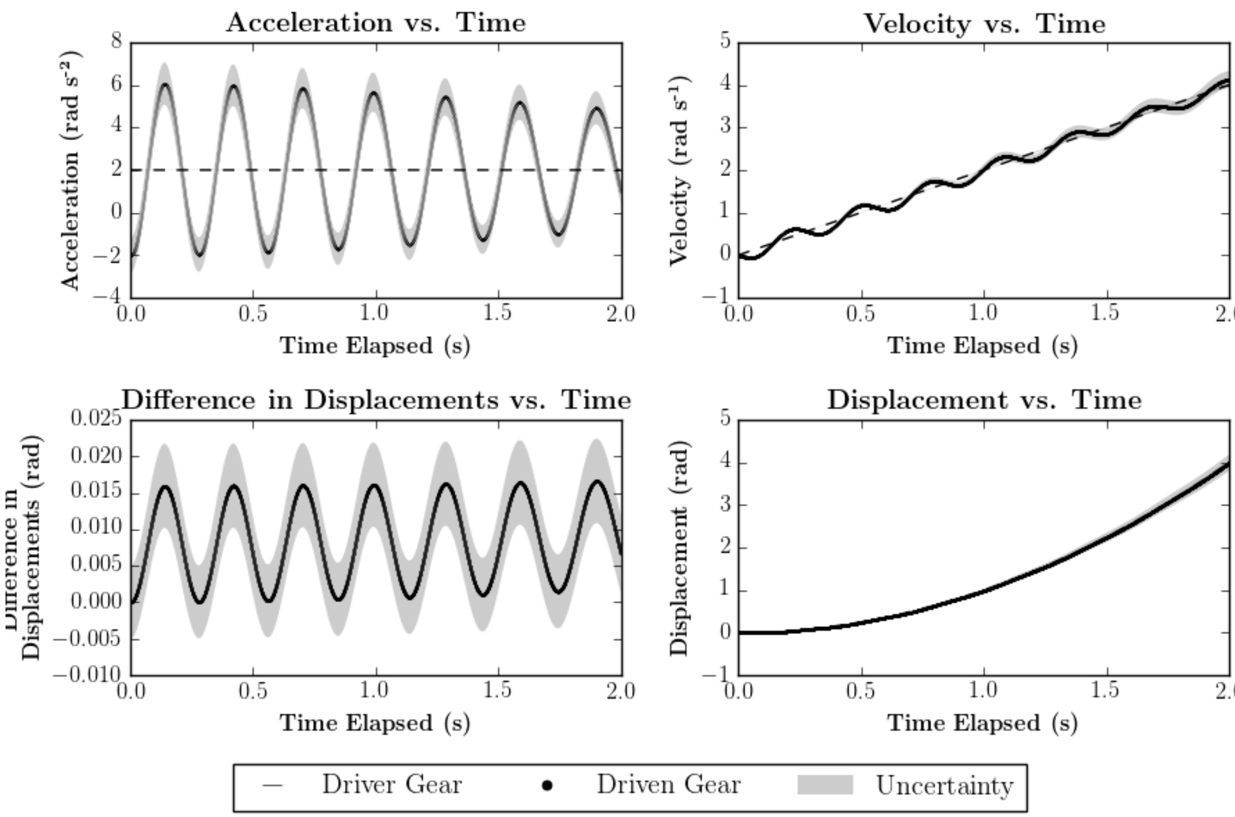
where $(2.00 \pm 0.02) \cdot 10^{-5}$ and $(5.00 \pm 0.05) \cdot 10^{-5}$ are arbitrary constants.

[21] This is one way of representing these resistive torques.

[22] The lowercase Greek letter *iota* (ι) was chosen to represent the coefficients of the various powers of $\Delta\dot{\vec{\theta}}_o$ because the uppercase letter I is conventionally used to represent a moment of inertia.

We will begin by looking at the behaviour of the system for the first two seconds of acceleration when the driver acceleration $\vec{\alpha}_i$ to $2.00\,\mathrm{rad\,s^{-2}} \pm 0.01\,\mathrm{rad\,s^{-2}}$. The results of the simulation are shown in Figure 15[23].

Figure 15: Runge-Kutta Program Output Given $\vec{\alpha}_i = 2.00\,\mathrm{rad\,s^{-2}} \pm 0.01\,\mathrm{rad\,s^{-2}}$ and $h = 0.0001\,\mathrm{s}$ for First $2\,\mathrm{s}$ of Acceleration

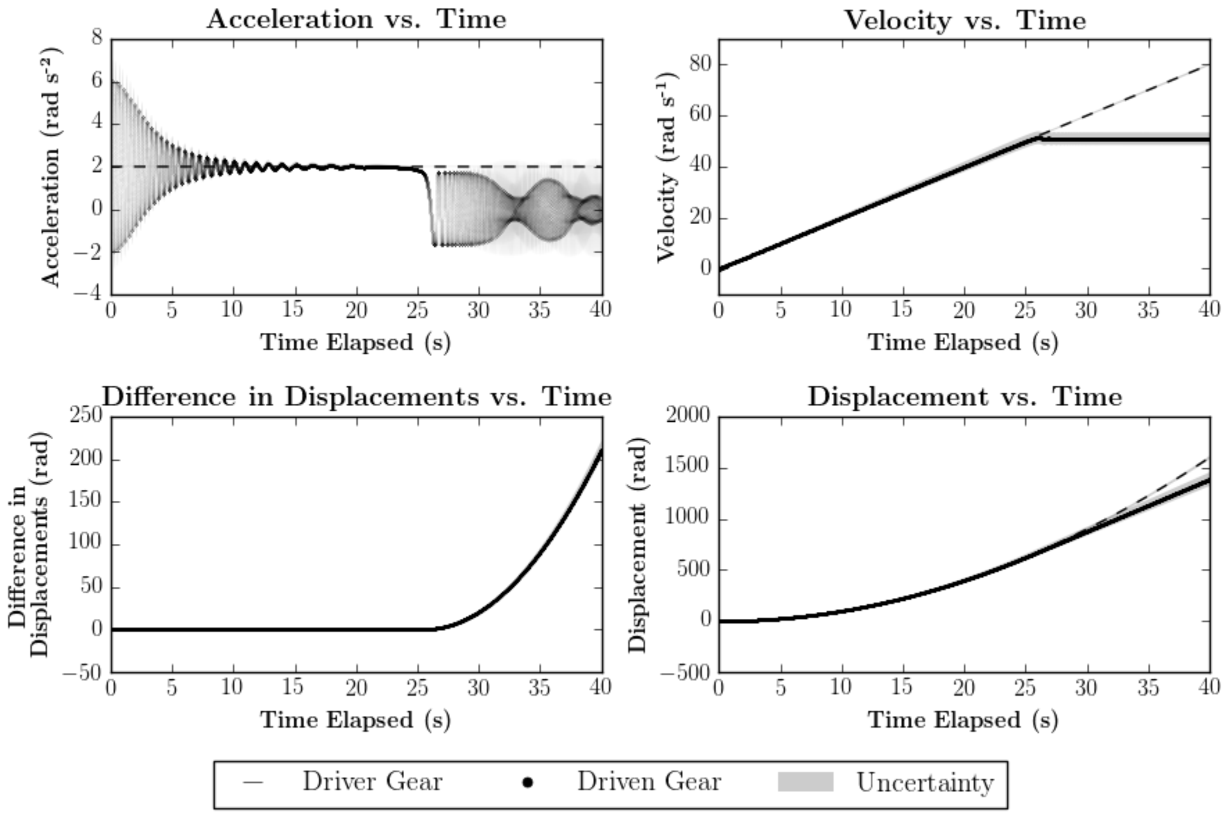


At first glance, it appears as though the rate of acceleration of this system is sustainable. However, because the moment of inertia of the gear now increases as the driven gear accelerates, there must be a certain amount of time after which the system begins to slip. To demonstrate this, we can predict the behaviour of the same system after $40\,\mathrm{s}$ has elapsed. The results are shown in Figure 16[24].

[23] The command used to generate these plots is `python runge-kutta.py 2.00+-0.01 2 0.0001 0.00257+-0.00006 0.0000200+-0.0000002 0.0000500+-0.0000005`.

[24] The command used to generate these plots is `python runge-kutta.py 2.00+-0.01 40 0.001 0.00257+-0.00006 0.0000200+-0.0000002 0.0000500+-0.0000005`.

Figure 16: Runge-Kutta Program Output Given $\vec{\alpha}_i = 2.00\,\mathrm{rad\,s^{-2}} \pm 0.01\,\mathrm{rad\,s^{-2}}$ and $h = 0.001\,\mathrm{s}$ for First $40\,\mathrm{s}$ of Acceleration



Sure enough, after $26.55\,\mathrm{s} \pm 0.04\,\mathrm{s}$, the system begins to slip indefinitely.

In fact, given enough time to accelerate at a constant rate, *every* magnetic gear train will begin to slip when friction cannot be ignored. This is because the moment of inertia will eventually grow as the driven gear accelerates to a point where the driven gear is just too "heavy" to continue being accelerated by the driver gear.

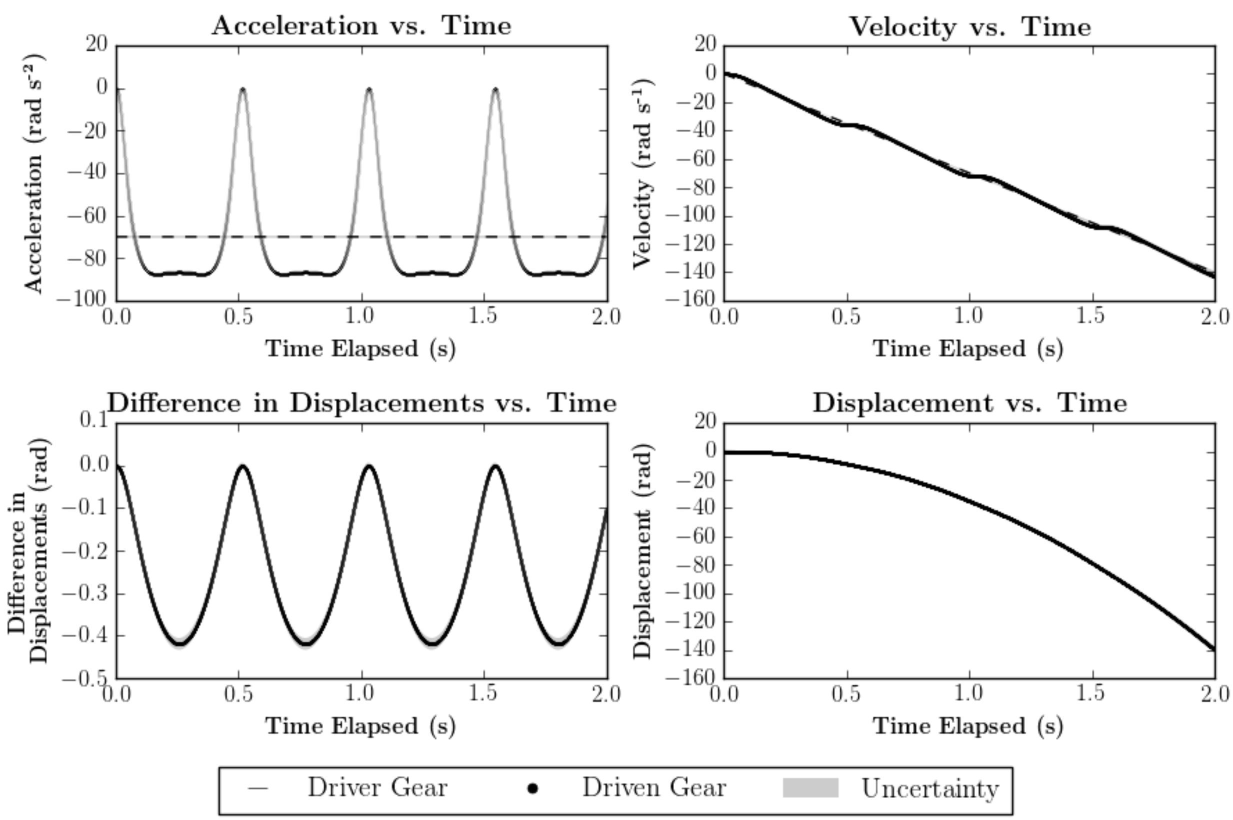
The acceleration-time plot in Figure 16 also suggests a certain degree of accuracy in the simulation: because the acceleration of the driven gear converges to the acceleration of the driver gear as it oscillates, it is clear that some kind of friction is at play. This is the same reason that a simple harmonic oscillator (i.e., a mass suspended by a spring) will eventually come to rest after being stretched out of equilibrium when not in a vacuum; the system is dampened by air friction (O'Neil, 1983).

H.2 Decelerating System

It is also possible for the driver gear to impose a negative torque on the driven gear should the displacement of the latter begin to exceed the displacement of the former. There is a maximum sustainable deceleration of the driver gear, and it should be equivalent to the maximum sustainable acceleration for a given gear train.

For example, given the same magnetic gear train as in Section ?? with a constant moment of inertia of $0.00257\,\mathrm{kg\,m^2} \pm 0.00006\,\mathrm{kg\,m^2}$, an acceleration of the driver gear of $-70.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ is sustainable, whereas an acceleration of $-80\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ is not. This is demonstrated in Figures 17[25] and 18[26], respectively.

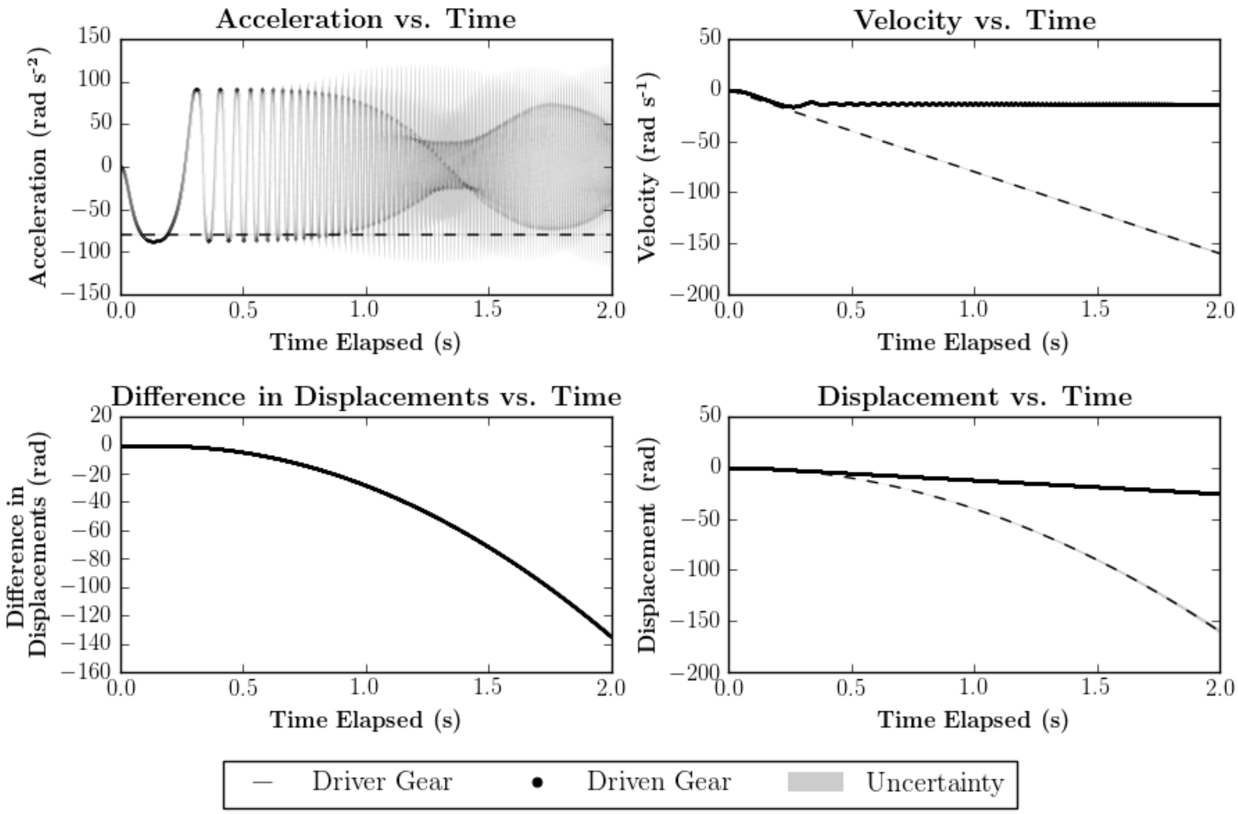
Figure 17: Runge-Kutta Program Output Given $\vec{\alpha}_i = -70.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ and $h = 0.0001\,\mathrm{s}$ for First 2 s of Acceleration



[25]The command used to generate these plots is `python runge-kutta.py -70.0+-0.5 2 0.0001 0.00257+-0.00006 0+-0 0+-0`.

[26]The command used to generate these plots is `python runge-kutta.py -80.0+-0.5 2 0.0001 0.00257+-0.00006 0+-0 0+-0`.

Figure 18: Runge-Kutta Program Output Given $\vec{\alpha}_i = -80.0\,\mathrm{rad\,s^{-2}} \pm 0.5\,\mathrm{rad\,s^{-2}}$ and $h = 0.0001\,\mathrm{s}$ for First $2\,\mathrm{s}$ of Acceleration



I Full Raw Dataset for Torque Curve Generation

Table 8: Full Raw Dataset of Torque Curve Generation for Trial 1

Data Point #	$\Delta\vec{\theta}_i$ (° ± 0.25 °)	$\Delta\vec{\theta}_o$ (° ± 0.25 °)	Left Scale Reading (g ± 0.01 g)	Right Scale Reading (g ± 0.01 g)
1	0.00	0.00	0.00	0.00
2	0.25	0.00	0.00	0.11
3	0.25	0.00	0.00	0.41
4	0.25	0.00	0.00	0.88
5	0.50	0.00	0.00	1.50
6	0.50	0.00	0.00	2.18
7	0.75	0.00	0.00	2.87
8	0.75	0.00	0.00	3.64
9	1.00	0.00	0.00	4.80
10	1.00	0.00	0.00	5.40
11	1.50	0.00	0.00	6.97
12	1.50	0.00	0.00	7.86
13	1.75	0.00	0.00	8.58
14	1.75	0.00	0.00	9.38
15	2.00	0.00	0.00	10.47
16	2.00	0.00	0.00	11.07
17	2.25	0.00	0.00	12.01
18	2.50	0.00	0.00	12.83
19	2.50	0.00	0.00	13.78
20	2.75	0.00	0.00	14.41
21	2.75	0.00	0.00	15.43
22	3.00	0.00	0.00	15.74
23	3.00	0.00	0.00	16.51
24	3.25	0.00	0.00	16.97
25	3.25	0.00	0.00	17.78
26	3.25	0.00	0.00	17.92
27	3.75	0.00	0.00	20.37
28	4.25	0.00	0.00	23.70
29	4.75	0.00	0.00	25.74
30	5.75	0.00	0.00	30.23
31	6.25	0.00	0.00	32.48
32	6.50	0.00	0.00	34.59
33	7.00	0.00	0.00	35.95
34	7.50	0.00	0.00	38.28
35	7.75	0.00	0.00	39.81
36	8.25	0.00	0.00	41.42
37	8.75	0.00	0.00	43.34
38	9.25	0.00	0.00	44.86
39	9.75	0.00	0.00	46.15
40	10.00	0.00	0.00	47.71
41	10.50	0.00	0.00	48.88
42	10.75	0.00	0.00	49.62
43	11.25	0.00	0.00	50.35
44	11.50	0.00	0.00	51.33
45	12.25	0.00	0.00	52.53
46	13.00	0.00	0.00	53.77
47	14.00	0.00	0.00	55.11
48	14.50	0.00	0.00	55.91
49	14.75	0.00	0.00	56.12
50	15.00	0.00	0.00	56.09
51	15.25	0.00	0.00	56.13
52	15.25	0.00	0.00	56.03
53	15.50	0.00	0.00	56.24
54	15.75	0.00	0.00	56.30
55	16.00	0.00	0.00	56.58
56	16.25	0.00	0.00	56.80
57	16.75	0.00	0.00	57.07
58	17.25	0.00	0.00	57.34
59	17.75	0.00	0.00	57.45
60	18.00	0.00	0.00	57.68
61	18.50	0.00	0.00	57.88
62	19.25	0.00	0.00	57.83
63	20.00	0.00	0.00	58.16
64	20.50	0.00	0.00	57.93
65	21.00	0.00	0.00	58.13
66	21.25	0.00	0.00	58.05
67	21.75	0.00	0.00	58.05

Table 8: Full Raw Dataset of Torque Curve Generation for Trial 1

Data Point #	$\Delta\vec{\theta}_i$ (° ± 0.25 °)	$\Delta\vec{\theta}_o$ (° ± 0.25 °)	**Left Scale Reading** (g ± 0.01 g)	**Right Scale Reading** (g ± 0.01 g)
68	22.00	0.00	0.00	57.90
69	22.25	0.00	0.00	57.81
70	22.75	0.00	0.00	57.84
71	23.00	0.00	0.00	57.81
72	23.25	0.00	0.00	57.78
73	23.75	0.00	0.00	57.76
74	24.25	0.00	0.00	57.71
75	24.50	0.00	0.00	57.58
76	24.75	0.00	0.00	57.48
77	25.25	0.00	0.00	57.30
78	25.50	0.00	0.00	57.19
79	25.75	0.00	0.00	56.98
80	26.00	0.00	0.00	56.74
81	26.25	0.00	0.00	56.43
82	26.50	0.00	0.00	55.94
83	26.75	0.00	0.00	55.77
84	27.00	0.00	0.00	55.78
85	27.25	0.00	0.00	55.55
86	27.50	0.00	0.00	55.50
87	28.25	0.00	0.00	55.38
88	29.25	0.00	0.00	54.00
89	29.50	0.00	0.00	52.36
90	30.50	0.00	0.00	52.13
91	30.75	0.00	0.00	50.52
92	31.50	0.00	0.00	49.50
93	32.25	0.00	0.00	47.31
94	32.75	0.00	0.00	45.68
95	33.25	0.00	0.00	43.57
96	33.75	0.00	0.00	41.00
97	34.25	0.00	0.00	38.12
98	35.50	0.00	0.00	34.63
99	35.75	0.00	0.00	32.84
100	36.25	0.00	0.00	30.84
101	37.00	0.00	0.00	28.35
102	37.25	0.00	0.00	26.26
103	37.75	0.00	0.00	24.30
104	38.25	0.00	0.00	22.65
105	38.50	0.00	0.00	21.20
106	39.00	0.00	0.00	19.47
107	39.25	0.00	0.00	17.36
108	39.75	0.00	0.00	15.27
109	40.25	0.00	0.00	12.64
110	40.50	0.00	0.00	11.11
111	41.00	0.00	0.00	8.60
112	41.50	0.00	0.00	5.90
113	42.25	0.00	0.00	2.32
114	42.25	0.00	0.00	1.63
115	42.75	0.00	0.85	0.00
116	42.75	0.00	1.91	0.00
117	43.25	0.00	3.52	0.00
118	43.50	0.00	5.34	0.00
119	44.00	0.00	7.92	0.00
120	44.25	0.00	9.03	0.00
121	44.50	0.00	10.13	0.00
122	44.50	0.00	11.25	0.00
123	45.00	0.00	12.82	0.00
124	45.25	0.00	13.76	0.00
125	46.00	0.00	17.43	0.00
126	46.00	0.00	18.42	0.00
127	46.50	0.00	21.14	0.00
128	47.25	0.00	23.91	0.00
129	47.75	0.00	26.51	0.00
130	48.50	0.00	29.70	0.00
131	49.00	0.00	31.57	0.00
132	49.50	0.00	33.66	0.00
133	50.25	0.00	36.76	0.00
134	50.75	0.00	38.15	0.00
135	51.25	0.00	38.77	0.00
136	52.00	0.00	40.93	0.00
137	52.75	0.00	43.15	0.00
138	53.50	0.00	44.83	0.00
139	54.25	0.00	46.91	0.00
140	54.75	0.00	48.08	0.00
141	55.50	0.00	49.33	0.00
142	56.00	0.00	50.54	0.00
143	56.50	0.00	51.31	0.00
144	56.50	0.00	52.11	0.00
145	56.75	0.00	52.21	0.00
146	57.50	0.00	52.99	0.00
147	58.25	0.00	53.91	0.00
148	59.25	0.00	54.50	0.00
149	60.50	0.00	54.93	0.00
150	61.00	0.00	54.89	0.00

Table 8: Full Raw Dataset of Torque Curve Generation for Trial 1

Data Point #	$\Delta\vec{\theta}_i$ (° ± 0.25 °)	$\Delta\vec{\theta}_o$ (° ± 0.25 °)	Left Scale Reading (g ± 0.01 g)	Right Scale Reading (g ± 0.01 g)
151	61.75	0.00	55.11	0.00
152	62.25	0.00	55.19	0.00
153	63.00	0.00	55.05	0.00
154	63.25	0.00	55.34	0.00
155	63.75	0.00	55.14	0.00
156	65.50	0.00	55.19	0.00
157	67.00	0.00	54.87	0.00
158	68.50	0.00	54.39	0.00
159	69.00	0.00	54.06	0.00
160	70.25	0.00	53.56	0.00
161	71.00	0.00	52.96	0.00
162	72.00	0.00	52.24	0.00
163	72.50	0.00	51.67	0.00
164	73.75	0.00	50.62	0.00
165	74.00	0.00	50.18	0.00
166	76.50	0.00	46.20	0.00
167	77.25	0.00	44.59	0.00
168	77.75	0.00	43.24	0.00
169	78.25	0.00	41.73	0.00
170	78.75	0.00	40.36	0.00
171	79.25	0.00	38.44	0.00
172	79.75	0.00	36.99	0.00
173	80.50	0.00	33.42	0.00
174	81.25	0.00	31.15	0.00
175	82.00	0.00	26.99	0.00
176	82.50	0.00	24.75	0.00
177	83.00	0.00	22.82	0.00

Table 9: Full Raw Dataset of Torque Curve Generation for Trial 2

Data Point #	$\Delta\vec{\theta}_i$ ($^\circ \pm 0.25^\circ$)	$\Delta\vec{\theta}_o$ ($^\circ \pm 0.25^\circ$)	Left Scale Reading (g $\pm$ 0.01 g)	Right Scale Reading (g $\pm$ 0.01 g)
1	0.00	0.00	0.00	0.00
2	0.25	0.00	0.00	0.14
3	0.25	0.00	0.00	0.51
4	0.50	0.00	0.00	0.98
5	0.75	0.00	0.00	1.55
6	0.75	0.00	0.00	2.34
7	1.00	0.00	0.00	3.05
8	1.00	0.00	0.00	3.67
9	1.00	0.00	0.00	4.86
10	1.25	0.00	0.00	5.41
11	1.50	0.00	0.00	7.09
12	1.50	0.00	0.00	7.95
13	1.75	0.00	0.00	8.67
14	1.75	0.00	0.00	9.41
15	2.00	0.00	0.00	10.58
16	2.00	0.00	0.00	11.16
17	2.25	0.00	0.00	12.15
18	2.75	0.00	0.00	12.93
19	2.75	0.00	0.00	13.86
20	2.75	0.00	0.00	14.45
21	2.75	0.00	0.00	15.56
22	3.00	0.00	0.00	15.85
23	3.00	0.00	0.00	16.62
24	3.25	0.00	0.00	17.15
25	3.50	0.00	0.00	17.97
26	3.50	0.00	0.00	18.08
27	4.00	0.00	0.00	20.44
28	4.25	0.00	0.00	23.71
29	4.75	0.00	0.00	25.82
30	5.75	0.00	0.00	30.29
31	6.25	0.00	0.00	32.65
32	6.50	0.00	0.00	34.75
33	7.00	0.00	0.00	36.01
34	7.75	0.00	0.00	38.46
35	7.75	0.00	0.00	39.89
36	8.25	0.00	0.00	41.49
37	9.00	0.00	0.00	43.46
38	9.25	0.00	0.00	44.93
39	9.75	0.00	0.00	46.35
40	10.00	0.00	0.00	47.89
41	10.50	0.00	0.00	48.93
42	10.75	0.00	0.00	49.80
43	11.25	0.00	0.00	50.42
44	11.50	0.00	0.00	51.36
45	12.25	0.00	0.00	52.71
46	13.00	0.00	0.00	53.81
47	14.00	0.00	0.00	55.15
48	14.50	0.00	0.00	56.11
49	15.00	0.00	0.00	56.18
50	15.00	0.00	0.00	56.22
51	15.25	0.00	0.00	56.27
52	15.25	0.00	0.00	56.06
53	15.75	0.00	0.00	56.43
54	15.75	0.00	0.00	56.38
55	16.00	0.00	0.00	56.59
56	16.25	0.00	0.00	56.85
57	16.75	0.00	0.00	57.09
58	17.50	0.00	0.00	57.44
59	17.75	0.00	0.00	57.58
60	18.25	0.00	0.00	57.81
61	18.50	0.00	0.00	58.04
62	19.50	0.00	0.00	57.95
63	20.00	0.00	0.00	58.21
64	20.50	0.00	0.00	58.04
65	21.00	0.00	0.00	58.31
66	21.25	0.00	0.00	58.12
67	21.75	0.00	0.00	58.23
68	22.00	0.00	0.00	58.06
69	22.25	0.00	0.00	57.86
70	22.75	0.00	0.00	57.95
71	23.00	0.00	0.00	57.96
72	23.25	0.00	0.00	57.92
73	24.00	0.00	0.00	57.88
74	24.25	0.00	0.00	57.89
75	24.50	0.00	0.00	57.73
76	25.00	0.00	0.00	57.54
77	25.25	0.00	0.00	57.38
78	25.50	0.00	0.00	57.20
79	25.75	0.00	0.00	57.12
80	26.25	0.00	0.00	56.80
81	26.25	0.00	0.00	56.48
82	26.50	0.00	0.00	56.02
83	27.00	0.00	0.00	55.84

Table 9: Full Raw Dataset of Torque Curve Generation for Trial 2

Data Point #	$\Delta\vec{\theta}_i$ (° ± 0.25 °)	$\Delta\vec{\theta}_o$ (° ± 0.25 °)	Left Scale Reading (g ± 0.01 g)	Right Scale Reading (g ± 0.01 g)
84	27.00	0.00	0.00	55.86
85	27.25	0.00	0.00	55.63
86	27.75	0.00	0.00	55.55
87	28.25	0.00	0.00	55.52
88	29.25	0.00	0.00	54.18
89	29.75	0.00	0.00	52.50
90	30.50	0.00	0.00	52.20
91	30.75	0.00	0.00	50.56
92	31.75	0.00	0.00	49.62
93	32.25	0.00	0.00	47.35
94	32.75	0.00	0.00	45.73
95	33.25	0.00	0.00	43.69
96	33.75	0.00	0.00	41.18
97	34.25	0.00	0.00	38.19
98	35.50	0.00	0.00	34.69
99	35.75	0.00	0.00	32.92
100	36.25	0.00	0.00	30.91
101	37.00	0.00	0.00	28.53
102	37.25	0.00	0.00	26.31
103	38.00	0.00	0.00	24.38
104	38.25	0.00	0.00	22.78
105	38.50	0.00	0.00	21.30
106	39.00	0.00	0.00	19.54
107	39.25	0.00	0.00	17.41
108	39.75	0.00	0.00	15.42
109	40.25	0.00	0.00	12.71
110	40.75	0.00	0.00	11.29
111	41.25	0.00	0.00	8.69
112	41.50	0.00	0.00	6.00
113	42.25	0.00	0.00	2.45
114	42.25	0.00	0.00	1.77
115	42.75	0.00	0.00	0.71
116	43.00	0.00	0.00	0.23
117	43.25	0.00	0.87	0.00
118	43.50	0.00	1.93	0.00
119	44.25	0.00	3.53	0.00
120	44.25	0.00	5.37	0.00
121	44.50	0.00	8.02	0.00
122	44.50	0.00	9.21	0.00
123	45.00	0.00	10.14	0.00
124	45.25	0.00	11.38	0.00
125	46.00	0.00	12.97	0.00
126	46.25	0.00	13.90	0.00
127	46.75	0.00	17.56	0.00
128	47.25	0.00	18.43	0.00
129	47.75	0.00	21.29	0.00
130	48.75	0.00	24.07	0.00
131	49.00	0.00	26.65	0.00
132	49.50	0.00	29.77	0.00
133	50.25	0.00	31.59	0.00
134	50.75	0.00	33.69	0.00
135	51.25	0.00	36.91	0.00
136	52.00	0.00	38.24	0.00
137	53.00	0.00	38.88	0.00
138	53.75	0.00	41.08	0.00
139	54.25	0.00	43.21	0.00
140	54.75	0.00	44.94	0.00
141	55.50	0.00	46.96	0.00
142	56.25	0.00	48.14	0.00
143	56.50	0.00	49.33	0.00
144	56.75	0.00	50.56	0.00
145	56.75	0.00	51.33	0.00
146	57.50	0.00	52.16	0.00
147	58.25	0.00	52.24	0.00
148	59.50	0.00	53.01	0.00
149	60.75	0.00	53.99	0.00
150	61.00	0.00	54.70	0.00
151	61.75	0.00	55.03	0.00
152	62.25	0.00	54.99	0.00
153	63.00	0.00	55.26	0.00
154	63.25	0.00	55.36	0.00
155	63.75	0.00	55.25	0.00
156	65.50	0.00	55.41	0.00
157	67.25	0.00	55.20	0.00
158	68.75	0.00	55.31	0.00
159	69.00	0.00	55.00	0.00
160	70.25	0.00	54.50	0.00
161	71.25	0.00	54.22	0.00
162	72.00	0.00	53.64	0.00
163	72.50	0.00	53.05	0.00
164	73.75	0.00	52.32	0.00
165	74.25	0.00	51.80	0.00
166	76.50	0.00	50.66	0.00

Table 9: Full Raw Dataset of Torque Curve Generation for Trial 2

Data Point #	$\mathbf{\Delta\vec{\theta}_i}$ (° ± **0.25** °)	$\mathbf{\Delta\vec{\theta}_o}$ (° ± **0.25** °)	**Left Scale Reading** (**g** ± **0.01 g**)	**Right Scale Reading** (**g** ± **0.01 g**)
167	77.25	0.00	50.27	0.00
168	78.00	0.00	46.35	0.00
169	78.25	0.00	44.64	0.00
170	78.75	0.00	43.42	0.00
171	79.25	0.00	41.85	0.00
172	79.75	0.00	40.37	0.00
173	80.50	0.00	38.51	0.00
174	81.00	0.00	37.09	0.00
175	81.25	0.00	33.57	0.00
176	81.50	0.00	31.21	0.00

Table 10: Full Raw Dataset of Torque Curve Generation for Trial 3

Data Point #	$\Delta\vec{\theta}_i$ ($^\circ \pm 0.25\,^\circ$)	$\Delta\vec{\theta}_o$ ($^\circ \pm 0.25\,^\circ$)	Left Scale Reading (g $\pm$ 0.01 g)	Right Scale Reading (g $\pm$ 0.01 g)
1	0.00	0.00	0.00	0.00
2	0.25	0.00	0.00	0.17
3	0.25	0.00	0.00	0.49
4	0.50	0.00	0.00	0.88
5	0.75	0.00	0.00	1.51
6	0.75	0.00	0.00	2.26
7	1.00	0.00	0.00	3.16
8	1.00	0.00	0.00	3.70
9	1.00	0.00	0.00	4.98
10	1.25	0.00	0.00	5.59
11	1.50	0.00	0.00	7.20
12	1.75	0.00	0.00	8.02
13	2.00	0.00	0.00	8.82
14	2.00	0.00	0.00	9.56
15	2.00	0.00	0.00	10.73
16	2.25	0.00	0.00	11.21
17	2.50	0.00	0.00	12.18
18	2.75	0.00	0.00	13.04
19	2.75	0.00	0.00	13.98
20	3.00	0.00	0.00	14.54
21	3.00	0.00	0.00	15.68
22	3.25	0.00	0.00	16.02
23	3.25	0.00	0.00	16.80
24	3.50	0.00	0.00	17.19
25	3.50	0.00	0.00	18.16
26	3.50	0.00	0.00	18.15
27	4.00	0.00	0.00	20.53
28	4.25	0.00	0.00	23.82
29	5.00	0.00	0.00	25.98
30	6.00	0.00	0.00	30.47
31	6.25	0.00	0.00	32.80
32	6.50	0.00	0.00	34.77
33	7.00	0.00	0.00	36.01
34	7.75	0.00	0.00	38.64
35	8.00	0.00	0.00	39.98
36	8.50	0.00	0.00	41.52
37	9.00	0.00	0.00	43.50
38	9.50	0.00	0.00	45.10
39	10.00	0.00	0.00	46.42
40	10.00	0.00	0.00	47.92
41	10.50	0.00	0.00	49.05
42	11.00	0.00	0.00	49.90
43	11.50	0.00	0.00	50.56
44	11.50	0.00	0.00	51.50
45	12.50	0.00	0.00	52.83
46	13.25	0.00	0.00	53.86
47	14.00	0.00	0.00	55.16
48	14.75	0.00	0.00	56.16
49	15.00	0.00	0.00	56.30
50	15.00	0.00	0.00	56.32
51	15.50	0.00	0.00	56.36
52	15.50	0.00	0.00	56.19
53	16.00	0.00	0.00	56.52
54	15.75	0.00	0.00	56.52
55	16.00	0.00	0.00	56.68
56	16.50	0.00	0.00	56.87
57	17.00	0.00	0.00	57.28
58	17.50	0.00	0.00	57.63
59	17.75	0.00	0.00	57.73
60	18.50	0.00	0.00	58.00
61	18.50	0.00	0.00	58.05
62	19.50	0.00	0.00	58.10
63	20.25	0.00	0.00	58.21
64	20.50	0.00	0.00	58.06
65	21.00	0.00	0.00	58.35
66	21.25	0.00	0.00	58.29
67	22.00	0.00	0.00	58.24
68	22.25	0.00	0.00	58.13
69	22.50	0.00	0.00	58.04
70	22.75	0.00	0.00	58.00
71	23.00	0.00	0.00	57.98
72	23.25	0.00	0.00	57.95
73	24.00	0.00	0.00	58.06
74	24.50	0.00	0.00	58.09
75	24.50	0.00	0.00	57.86
76	25.00	0.00	0.00	57.59
77	25.25	0.00	0.00	57.41
78	25.50	0.00	0.00	57.20
79	25.75	0.00	0.00	57.25
80	26.25	0.00	0.00	56.80
81	26.50	0.00	0.00	56.62
82	26.50	0.00	0.00	56.18
83	27.00	0.00	0.00	55.98

Table 10: Full Raw Dataset of Torque Curve Generation for Trial 3

Data Point #	$\Delta\vec{\theta}_i$ ($^\circ \pm 0.25^\circ$)	$\Delta\vec{\theta}_o$ ($^\circ \pm 0.25^\circ$)	Left Scale Reading (g $\pm$ 0.01 g)	Right Scale Reading (g $\pm$ 0.01 g)
84	27.00	0.00	0.00	55.94
85	27.50	0.00	0.00	55.76
86	27.75	0.00	0.00	55.57
87	28.50	0.00	0.00	55.58
88	29.50	0.00	0.00	54.36
89	29.75	0.00	0.00	52.69
90	30.50	0.00	0.00	52.30
91	31.00	0.00	0.00	50.64
92	32.00	0.00	0.00	49.73
93	32.25	0.00	0.00	47.44
94	33.00	0.00	0.00	45.80
95	33.25	0.00	0.00	43.83
96	33.75	0.00	0.00	41.26
97	34.25	0.00	0.00	38.25
98	35.75	0.00	0.00	34.72
99	35.75	0.00	0.00	33.08
100	36.50	0.00	0.00	31.09
101	37.00	0.00	0.00	28.62
102	37.50	0.00	0.00	26.39
103	38.00	0.00	0.00	24.48
104	38.25	0.00	0.00	22.82
105	38.50	0.00	0.00	21.43
106	39.00	0.00	0.00	19.63
107	39.50	0.00	0.00	17.54
108	40.00	0.00	0.00	15.52
109	40.50	0.00	0.00	12.74
110	41.00	0.00	0.00	11.32
111	41.25	0.00	0.00	8.87
112	41.50	0.00	0.00	6.17
113	42.50	0.00	0.00	2.62
114	42.50	0.00	0.00	1.79
115	43.00	0.00	0.00	0.83
116	43.25	0.00	0.94	0.00
117	43.25	0.00	2.05	0.00
118	43.50	0.00	3.54	0.00
119	44.50	0.00	5.48	0.00
120	44.25	0.00	7.99	0.00
121	44.50	0.00	9.05	0.00
122	44.75	0.00	10.24	0.00
123	45.00	0.00	11.44	0.00
124	45.25	0.00	12.92	0.00
125	46.00	0.00	13.80	0.00
126	46.25	0.00	17.50	0.00
127	47.00	0.00	18.45	0.00
128	47.50	0.00	21.16	0.00
129	47.75	0.00	24.05	0.00
130	48.75	0.00	26.55	0.00
131	49.25	0.00	29.79	0.00
132	49.50	0.00	31.70	0.00
133	50.50	0.00	33.73	0.00
134	51.00	0.00	36.92	0.00
135	51.25	0.00	38.26	0.00
136	52.00	0.00	38.88	0.00
137	53.25	0.00	41.05	0.00
138	54.00	0.00	43.20	0.00
139	54.50	0.00	44.94	0.00
140	55.00	0.00	46.93	0.00
141	55.50	0.00	48.17	0.00
142	56.25	0.00	49.52	0.00
143	56.50	0.00	50.66	0.00
144	56.75	0.00	51.51	0.00
145	56.75	0.00	52.27	0.00
146	57.50	0.00	52.37	0.00
147	58.25	0.00	53.05	0.00
148	59.50	0.00	54.10	0.00
149	60.75	0.00	54.60	0.00
150	61.00	0.00	54.96	0.00
151	61.75	0.00	55.09	0.00
152	62.50	0.00	55.28	0.00
153	63.00	0.00	55.31	0.00
154	63.50	0.00	55.22	0.00
155	63.75	0.00	55.43	0.00
156	65.75	0.00	55.14	0.00
157	67.25	0.00	55.36	0.00
158	68.75	0.00	55.02	0.00
159	69.00	0.00	54.46	0.00
160	70.25	0.00	54.17	0.00
161	71.25	0.00	53.73	0.00
162	72.25	0.00	53.12	0.00
163	72.50	0.00	52.36	0.00
164	73.75	0.00	51.78	0.00
165	74.25	0.00	50.66	0.00
166	76.50	0.00	50.21	0.00

Table 10: Full Raw Dataset of Torque Curve Generation for Trial 3

Data Point #	$\boldsymbol{\Delta\vec{\theta}_i}$ ($^\circ \pm$ **0.25** $^\circ$)	$\boldsymbol{\Delta\vec{\theta}_o}$ ($^\circ \pm$ **0.25** $^\circ$)	**Left Scale Reading (g $\pm$ 0.01 g)**	**Right Scale Reading (g $\pm$ 0.01 g)**
167	77.50	0.00	46.23	0.00
168	78.25	0.00	44.71	0.00
169	78.50	0.00	43.36	0.00
170	78.75	0.00	41.79	0.00
171	79.50	0.00	40.40	0.00
172	79.75	0.00	38.51	0.00
173	80.50	0.00	37.15	0.00
174	81.50	0.00	33.42	0.00
175	82.00	0.00	31.32	0.00
176	82.50	0.00	27.01	0.00

J Full Processed Dataset for Torque Curve Generation

Table 11: Full Processed Dataset of Torque Curve Generation for Trial 1

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	Torque (N m)	±	Torque Unc. (N m)
1	0.000	0	±	0
2	0.004	0.00045	±	0.00005
3	0.004	0.00166	±	0.00007
4	0.004	0.0036	±	0.0001
5	0.009	0.0061	±	0.0001
6	0.009	0.0089	±	0.0002
7	0.013	0.0117	±	0.0002
8	0.013	0.0148	±	0.0003
9	0.017	0.0195	±	0.0003
10	0.017	0.0219	±	0.0004
11	0.026	0.0283	±	0.0005
12	0.026	0.0319	±	0.0005
13	0.031	0.0348	±	0.0006
14	0.031	0.0381	±	0.0006
15	0.035	0.0425	±	0.0007
16	0.035	0.0449	±	0.0007
17	0.039	0.0488	±	0.0008
18	0.044	0.0521	±	0.0008
19	0.044	0.0559	±	0.0009
20	0.048	0.0585	±	0.0009
21	0.048	0.063	±	0.001
22	0.052	0.064	±	0.001
23	0.052	0.067	±	0.001
24	0.057	0.069	±	0.001
25	0.057	0.072	±	0.001
26	0.057	0.073	±	0.001
27	0.065	0.083	±	0.001
28	0.074	0.096	±	0.002
29	0.083	0.105	±	0.002
30	0.100	0.123	±	0.002
31	0.109	0.132	±	0.002
32	0.113	0.140	±	0.002
33	0.122	0.146	±	0.002
34	0.131	0.155	±	0.002
35	0.135	0.162	±	0.003
36	0.144	0.168	±	0.003
37	0.153	0.176	±	0.003
38	0.161	0.182	±	0.003
39	0.170	0.187	±	0.003
40	0.175	0.194	±	0.003
41	0.183	0.198	±	0.003
42	0.188	0.201	±	0.003
43	0.196	0.204	±	0.003
44	0.201	0.208	±	0.003
45	0.214	0.213	±	0.003
46	0.227	0.218	±	0.003
47	0.244	0.224	±	0.003
48	0.253	0.227	±	0.004
49	0.257	0.228	±	0.004
50	0.262	0.228	±	0.004
51	0.266	0.228	±	0.004
52	0.266	0.227	±	0.004
53	0.271	0.228	±	0.004
54	0.275	0.229	±	0.004
55	0.279	0.230	±	0.004
56	0.284	0.231	±	0.004
57	0.292	0.232	±	0.004
58	0.301	0.233	±	0.004
59	0.310	0.233	±	0.004
60	0.314	0.234	±	0.004
61	0.323	0.235	±	0.004
62	0.336	0.235	±	0.004
63	0.349	0.236	±	0.004
64	0.358	0.235	±	0.004
65	0.367	0.236	±	0.004
66	0.371	0.236	±	0.004
67	0.380	0.236	±	0.004
68	0.384	0.235	±	0.004

Table 11: Full Processed Dataset of Torque Curve Generation for Trial 1

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	Torque (N m)	±	Torque Unc. (N m)
69	0.388	0.235	±	0.004
70	0.397	0.235	±	0.004
71	0.401	0.235	±	0.004
72	0.406	0.235	±	0.004
73	0.415	0.234	±	0.004
74	0.423	0.234	±	0.004
75	0.428	0.234	±	0.004
76	0.432	0.233	±	0.004
77	0.441	0.233	±	0.004
78	0.445	0.232	±	0.004
79	0.449	0.231	±	0.004
80	0.454	0.230	±	0.004
81	0.458	0.229	±	0.004
82	0.463	0.227	±	0.004
83	0.467	0.226	±	0.004
84	0.471	0.226	±	0.004
85	0.476	0.226	±	0.004
86	0.480	0.225	±	0.004
87	0.493	0.225	±	0.004
88	0.511	0.219	±	0.003
89	0.515	0.213	±	0.003
90	0.532	0.212	±	0.003
91	0.537	0.205	±	0.003
92	0.550	0.201	±	0.003
93	0.563	0.192	±	0.003
94	0.572	0.185	±	0.003
95	0.580	0.177	±	0.003
96	0.589	0.166	±	0.003
97	0.598	0.155	±	0.002
98	0.620	0.141	±	0.002
99	0.624	0.133	±	0.002
100	0.633	0.125	±	0.002
101	0.646	0.115	±	0.002
102	0.650	0.107	±	0.002
103	0.659	0.099	±	0.002
104	0.668	0.092	±	0.001
105	0.672	0.086	±	0.001
106	0.681	0.079	±	0.001
107	0.685	0.070	±	0.001
108	0.694	0.062	±	0.001
109	0.702	0.0513	±	0.0008
110	0.707	0.0451	±	0.0007
111	0.716	0.0349	±	0.0006
112	0.724	0.0240	±	0.0004
113	0.737	0.0094	±	0.0002
114	0.737	0.0066	±	0.0002
115	0.746	-0.00345	±	0.00009
116	0.746	-0.0078	±	0.0002
117	0.755	-0.0143	±	0.0003
118	0.759	-0.0217	±	0.0004
119	0.768	-0.0322	±	0.0005
120	0.772	-0.0367	±	0.0006
121	0.777	-0.0411	±	0.0007
122	0.777	-0.0457	±	0.0007
123	0.785	-0.0520	±	0.0008
124	0.790	-0.0559	±	0.0009
125	0.803	-0.071	±	0.001
126	0.803	-0.075	±	0.001
127	0.812	-0.086	±	0.001
128	0.825	-0.097	±	0.002
129	0.833	-0.108	±	0.002
130	0.846	-0.121	±	0.002
131	0.855	-0.128	±	0.002
132	0.864	-0.137	±	0.002
133	0.877	-0.149	±	0.002
134	0.886	-0.155	±	0.002
135	0.894	-0.157	±	0.002
136	0.908	-0.166	±	0.003
137	0.921	-0.175	±	0.003
138	0.934	-0.182	±	0.003
139	0.947	-0.190	±	0.003
140	0.956	-0.195	±	0.003
141	0.969	-0.200	±	0.003
142	0.977	-0.205	±	0.003
143	0.986	-0.208	±	0.003
144	0.986	-0.212	±	0.003
145	0.990	-0.212	±	0.003
146	1.004	-0.215	±	0.003
147	1.017	-0.219	±	0.003
148	1.034	-0.221	±	0.003
149	1.056	-0.223	±	0.003
150	1.065	-0.223	±	0.003
151	1.078	-0.224	±	0.003
152	1.086	-0.224	±	0.003

Table 11: Full Processed Dataset of Torque Curve Generation for Trial 1

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	Torque (N m)	±	Torque Unc. (N m)
153	1.100	-0.223	±	0.003
154	1.104	-0.225	±	0.004
155	1.113	-0.224	±	0.003
156	1.143	-0.224	±	0.003
157	1.169	-0.223	±	0.003
158	1.196	-0.221	±	0.003
159	1.204	-0.219	±	0.003
160	1.226	-0.217	±	0.003
161	1.239	-0.215	±	0.003
162	1.257	-0.212	±	0.003
163	1.265	-0.210	±	0.003
164	1.287	-0.206	±	0.003
165	1.292	-0.204	±	0.003
166	1.335	-0.188	±	0.003
167	1.348	-0.181	±	0.003
168	1.357	-0.176	±	0.003
169	1.366	-0.169	±	0.003
170	1.374	-0.164	±	0.003
171	1.383	-0.156	±	0.002
172	1.392	-0.150	±	0.002
173	1.405	-0.136	±	0.002
174	1.418	-0.126	±	0.002
175	1.431	-0.110	±	0.002
176	1.440	-0.100	±	0.002
177	1.449	-0.093	±	0.001

Table 12: Full Processed Dataset of Torque Curve Generation for Trial 2

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	Torque (N m)	±	Torque Unc. (N m)
1	0.000	0	±	0
2	0.004	0.00057	±	0.00005
3	0.004	0.00207	±	0.00007
4	0.009	0.0040	±	0.0001
5	0.013	0.0063	±	0.0001
6	0.013	0.0095	±	0.0002
7	0.017	0.0124	±	0.0002
8	0.017	0.0149	±	0.0003
9	0.017	0.0197	±	0.0003
10	0.022	0.0220	±	0.0004
11	0.026	0.0288	±	0.0005
12	0.026	0.0323	±	0.0005
13	0.031	0.0352	±	0.0006
14	0.031	0.0382	±	0.0006
15	0.035	0.0430	±	0.0007
16	0.035	0.0453	±	0.0007
17	0.039	0.0493	±	0.0008
18	0.048	0.0525	±	0.0008
19	0.048	0.0563	±	0.0009
20	0.048	0.0587	±	0.0009
21	0.048	0.063	±	0.001
22	0.052	0.064	±	0.001
23	0.052	0.067	±	0.001
24	0.057	0.070	±	0.001
25	0.061	0.073	±	0.001
26	0.061	0.073	±	0.001
27	0.070	0.083	±	0.001
28	0.074	0.096	±	0.002
29	0.083	0.105	±	0.002
30	0.100	0.123	±	0.002
31	0.109	0.133	±	0.002
32	0.113	0.141	±	0.002
33	0.122	0.146	±	0.002
34	0.135	0.156	±	0.002
35	0.135	0.162	±	0.003
36	0.144	0.168	±	0.003
37	0.157	0.176	±	0.003
38	0.161	0.182	±	0.003
39	0.170	0.188	±	0.003
40	0.175	0.194	±	0.003
41	0.183	0.199	±	0.003
42	0.188	0.202	±	0.003
43	0.196	0.205	±	0.003
44	0.201	0.209	±	0.003
45	0.214	0.214	±	0.003
46	0.227	0.218	±	0.003
47	0.244	0.224	±	0.003
48	0.253	0.228	±	0.004
49	0.262	0.228	±	0.004
50	0.262	0.228	±	0.004
51	0.266	0.228	±	0.004
52	0.266	0.228	±	0.004
53	0.275	0.229	±	0.004
54	0.275	0.229	±	0.004
55	0.279	0.230	±	0.004
56	0.284	0.231	±	0.004
57	0.292	0.232	±	0.004
58	0.305	0.233	±	0.004
59	0.310	0.234	±	0.004
60	0.319	0.235	±	0.004
61	0.323	0.236	±	0.004
62	0.340	0.235	±	0.004
63	0.349	0.236	±	0.004
64	0.358	0.236	±	0.004
65	0.367	0.237	±	0.004
66	0.371	0.236	±	0.004
67	0.380	0.236	±	0.004
68	0.384	0.236	±	0.004
69	0.388	0.235	±	0.004
70	0.397	0.235	±	0.004
71	0.401	0.235	±	0.004
72	0.406	0.235	±	0.004
73	0.419	0.235	±	0.004
74	0.423	0.235	±	0.004
75	0.428	0.234	±	0.004
76	0.436	0.234	±	0.004
77	0.441	0.233	±	0.004
78	0.445	0.232	±	0.004
79	0.449	0.232	±	0.004
80	0.458	0.231	±	0.004
81	0.458	0.229	±	0.004
82	0.463	0.227	±	0.004
83	0.471	0.227	±	0.004
84	0.471	0.227	±	0.004

Table 12: Full Processed Dataset of Torque Curve Generation for Trial 2

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	Torque (N m)	±	Torque Unc. (N m)
85	0.476	0.226	±	0.004
86	0.484	0.226	±	0.004
87	0.493	0.225	±	0.004
88	0.511	0.220	±	0.003
89	0.519	0.213	±	0.003
90	0.532	0.212	±	0.003
91	0.537	0.205	±	0.003
92	0.554	0.201	±	0.003
93	0.563	0.192	±	0.003
94	0.572	0.186	±	0.003
95	0.580	0.177	±	0.003
96	0.589	0.167	±	0.003
97	0.598	0.155	±	0.002
98	0.620	0.141	±	0.002
99	0.624	0.134	±	0.002
100	0.633	0.125	±	0.002
101	0.646	0.116	±	0.002
102	0.650	0.107	±	0.002
103	0.663	0.099	±	0.002
104	0.668	0.092	±	0.001
105	0.672	0.086	±	0.001
106	0.681	0.079	±	0.001
107	0.685	0.071	±	0.001
108	0.694	0.063	±	0.001
109	0.702	0.0516	±	0.0008
110	0.711	0.0458	±	0.0007
111	0.720	0.0353	±	0.0006
112	0.724	0.0244	±	0.0004
113	0.737	0.0099	±	0.0002
114	0.737	0.0072	±	0.0002
115	0.746	0.00288	±	0.00009
116	0.750	0.00093	±	0.00005
117	0.755	-0.0035	±	0.0001
118	0.759	-0.0078	±	0.0002
119	0.772	-0.0143	±	0.0003
120	0.772	-0.0218	±	0.0004
121	0.777	-0.0326	±	0.0005
122	0.777	-0.0374	±	0.0006
123	0.785	-0.0412	±	0.0007
124	0.790	-0.0462	±	0.0008
125	0.803	-0.0527	±	0.0009
126	0.807	-0.0564	±	0.0009
127	0.816	-0.071	±	0.001
128	0.825	-0.075	±	0.001
129	0.833	-0.086	±	0.001
130	0.851	-0.098	±	0.002
131	0.855	-0.108	±	0.002
132	0.864	-0.121	±	0.002
133	0.877	-0.128	±	0.002
134	0.886	-0.137	±	0.002
135	0.894	-0.150	±	0.002
136	0.908	-0.155	±	0.002
137	0.925	-0.158	±	0.002
138	0.938	-0.167	±	0.003
139	0.947	-0.175	±	0.003
140	0.956	-0.182	±	0.003
141	0.969	-0.191	±	0.003
142	0.982	-0.195	±	0.003
143	0.986	-0.200	±	0.003
144	0.990	-0.205	±	0.003
145	0.990	-0.208	±	0.003
146	1.004	-0.212	±	0.003
147	1.017	-0.212	±	0.003
148	1.038	-0.215	±	0.003
149	1.060	-0.219	±	0.003
150	1.065	-0.222	±	0.003
151	1.078	-0.223	±	0.003
152	1.086	-0.223	±	0.003
153	1.100	-0.224	±	0.003
154	1.104	-0.225	±	0.004
155	1.113	-0.224	±	0.003
156	1.143	-0.225	±	0.004
157	1.174	-0.224	±	0.003
158	1.200	-0.225	±	0.004
159	1.204	-0.223	±	0.003
160	1.226	-0.221	±	0.003
161	1.244	-0.220	±	0.003
162	1.257	-0.218	±	0.003
163	1.265	-0.215	±	0.003
164	1.287	-0.212	±	0.003
165	1.296	-0.210	±	0.003
166	1.335	-0.206	±	0.003
167	1.348	-0.204	±	0.003
168	1.361	-0.188	±	0.003

Table 12: Full Processed Dataset of Torque Curve Generation for Trial 2

Data Point #	$\Delta\vec{\Theta}$ (rad $\pm$ 0.008 rad)	Torque (N m)	$\pm$	Torque Unc. (N m)
169	1.366	-0.181	$\pm$	0.003
170	1.374	-0.176	$\pm$	0.003
171	1.383	-0.170	$\pm$	0.003
172	1.392	-0.164	$\pm$	0.003
173	1.405	-0.156	$\pm$	0.002
174	1.414	-0.151	$\pm$	0.002
175	1.418	-0.136	$\pm$	0.002
176	1.422	-0.127	$\pm$	0.002

Table 13: Full Processed Dataset of Torque Curve Generation for Trial 3

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	Torque (N m)	±	Torque Unc. (N m)
1	0.000	0	±	0
2	0.004	0.00069	±	0.00005
3	0.004	0.00199	±	0.00007
4	0.009	0.0036	±	0.0001
5	0.013	0.0061	±	0.0001
6	0.013	0.0092	±	0.0002
7	0.017	0.0128	±	0.0002
8	0.017	0.0150	±	0.0003
9	0.017	0.0202	±	0.0004
10	0.022	0.0227	±	0.0004
11	0.026	0.0292	±	0.0005
12	0.031	0.0326	±	0.0005
13	0.035	0.0358	±	0.0006
14	0.035	0.0388	±	0.0006
15	0.035	0.0436	±	0.0007
16	0.039	0.0455	±	0.0007
17	0.044	0.0494	±	0.0008
18	0.048	0.0529	±	0.0009
19	0.048	0.0568	±	0.0009
20	0.052	0.059	±	0.001
21	0.052	0.064	±	0.001
22	0.057	0.065	±	0.001
23	0.057	0.068	±	0.001
24	0.061	0.070	±	0.001
25	0.061	0.074	±	0.001
26	0.061	0.074	±	0.001
27	0.070	0.083	±	0.001
28	0.074	0.097	±	0.002
29	0.087	0.105	±	0.002
30	0.105	0.124	±	0.002
31	0.109	0.133	±	0.002
32	0.113	0.141	±	0.002
33	0.122	0.146	±	0.002
34	0.135	0.157	±	0.002
35	0.140	0.162	±	0.003
36	0.148	0.169	±	0.003
37	0.157	0.177	±	0.003
38	0.166	0.183	±	0.003
39	0.175	0.188	±	0.003
40	0.175	0.195	±	0.003
41	0.183	0.199	±	0.003
42	0.192	0.203	±	0.003
43	0.201	0.205	±	0.003
44	0.201	0.209	±	0.003
45	0.218	0.214	±	0.003
46	0.231	0.219	±	0.003
47	0.244	0.224	±	0.003
48	0.257	0.228	±	0.004
49	0.262	0.229	±	0.004
50	0.262	0.229	±	0.004
51	0.271	0.229	±	0.004
52	0.271	0.228	±	0.004
53	0.279	0.229	±	0.004
54	0.275	0.229	±	0.004
55	0.279	0.230	±	0.004
56	0.288	0.231	±	0.004
57	0.297	0.233	±	0.004
58	0.305	0.234	±	0.004
59	0.310	0.234	±	0.004
60	0.323	0.235	±	0.004
61	0.323	0.236	±	0.004
62	0.340	0.236	±	0.004
63	0.353	0.236	±	0.004
64	0.358	0.236	±	0.004
65	0.367	0.237	±	0.004
66	0.371	0.237	±	0.004
67	0.384	0.236	±	0.004
68	0.388	0.236	±	0.004
69	0.393	0.236	±	0.004
70	0.397	0.235	±	0.004
71	0.401	0.235	±	0.004
72	0.406	0.235	±	0.004
73	0.419	0.236	±	0.004
74	0.428	0.236	±	0.004
75	0.428	0.235	±	0.004
76	0.436	0.234	±	0.004
77	0.441	0.233	±	0.004
78	0.445	0.232	±	0.004
79	0.449	0.232	±	0.004
80	0.458	0.231	±	0.004
81	0.463	0.230	±	0.004
82	0.463	0.228	±	0.004
83	0.471	0.227	±	0.004
84	0.471	0.227	±	0.004

Table 13: Full Processed Dataset of Torque Curve Generation for Trial 3

Data Point #	$\Delta\vec{\Theta}$ (rad ± 0.008 rad)	Torque (N m)	±	Torque Unc. (N m)
85	0.480	0.226	±	0.004
86	0.484	0.226	±	0.004
87	0.497	0.226	±	0.004
88	0.515	0.221	±	0.003
89	0.519	0.214	±	0.003
90	0.532	0.212	±	0.003
91	0.541	0.206	±	0.003
92	0.559	0.202	±	0.003
93	0.563	0.193	±	0.003
94	0.576	0.186	±	0.003
95	0.580	0.178	±	0.003
96	0.589	0.168	±	0.003
97	0.598	0.155	±	0.002
98	0.624	0.141	±	0.002
99	0.624	0.134	±	0.002
100	0.637	0.126	±	0.002
101	0.646	0.116	±	0.002
102	0.654	0.107	±	0.002
103	0.663	0.099	±	0.002
104	0.668	0.093	±	0.001
105	0.672	0.087	±	0.001
106	0.681	0.080	±	0.001
107	0.689	0.071	±	0.001
108	0.698	0.063	±	0.001
109	0.707	0.0517	±	0.0008
110	0.716	0.0460	±	0.0007
111	0.720	0.0360	±	0.0006
112	0.724	0.0250	±	0.0004
113	0.742	0.0106	±	0.0002
114	0.742	0.0073	±	0.0002
115	0.750	0.00337	±	0.00009
116	0.755	-0.0038	±	0.0001
117	0.755	-0.0083	±	0.0002
118	0.759	-0.0144	±	0.0003
119	0.777	-0.0222	±	0.0004
120	0.772	-0.0324	±	0.0005
121	0.777	-0.0367	±	0.0006
122	0.781	-0.0416	±	0.0007
123	0.785	-0.0464	±	0.0008
124	0.790	-0.0525	±	0.0008
125	0.803	-0.0560	±	0.0009
126	0.807	-0.071	±	0.001
127	0.820	-0.075	±	0.001
128	0.829	-0.086	±	0.001
129	0.833	-0.098	±	0.002
130	0.851	-0.108	±	0.002
131	0.860	-0.121	±	0.002
132	0.864	-0.129	±	0.002
133	0.881	-0.137	±	0.002
134	0.890	-0.150	±	0.002
135	0.894	-0.155	±	0.002
136	0.908	-0.158	±	0.002
137	0.929	-0.167	±	0.003
138	0.942	-0.175	±	0.003
139	0.951	-0.182	±	0.003
140	0.960	-0.191	±	0.003
141	0.969	-0.196	±	0.003
142	0.982	-0.201	±	0.003
143	0.986	-0.206	±	0.003
144	0.990	-0.209	±	0.003
145	0.990	-0.212	±	0.003
146	1.004	-0.213	±	0.003
147	1.017	-0.215	±	0.003
148	1.038	-0.220	±	0.003
149	1.060	-0.222	±	0.003
150	1.065	-0.223	±	0.003
151	1.078	-0.224	±	0.003
152	1.091	-0.224	±	0.003
153	1.100	-0.225	±	0.004
154	1.108	-0.224	±	0.003
155	1.113	-0.225	±	0.004
156	1.148	-0.224	±	0.003
157	1.174	-0.225	±	0.004
158	1.200	-0.223	±	0.003
159	1.204	-0.221	±	0.003
160	1.226	-0.220	±	0.003
161	1.244	-0.218	±	0.003
162	1.261	-0.216	±	0.003
163	1.265	-0.213	±	0.003
164	1.287	-0.210	±	0.003
165	1.296	-0.206	±	0.003
166	1.335	-0.204	±	0.003
167	1.353	-0.188	±	0.003
168	1.366	-0.182	±	0.003

Table 13: Full Processed Dataset of Torque Curve Generation for Trial 3

Data Point #	$\Delta\vec{\Theta}$ (rad $\pm$ 0.008 rad)	Torque (N m)	$\pm$	Torque Unc. (N m)
169	1.370	-0.176	$\pm$	0.003
170	1.374	-0.170	$\pm$	0.003
171	1.388	-0.164	$\pm$	0.003
172	1.392	-0.156	$\pm$	0.002
173	1.405	-0.151	$\pm$	0.002
174	1.422	-0.136	$\pm$	0.002
175	1.431	-0.127	$\pm$	0.002
176	1.440	-0.110	$\pm$	0.002